BULLETIN OF THE ASTRONOMICAL INSTITUTES OF THE NETHERLANDS.

1933 December 13 Volume VII. No. 252

COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

Note on the energy equation, by *J. Woltjer Jr.*

Some years ago ROSSELAND [1]) has published an extensive treatment of the equation expressing conservation of energy of matter moving in a field of radiation. An essential feature of this author's procedure consists in the examination of what happens to the energy contained in a *fixed* element of volume. Hence the resulting equation contains the flux of radiation referred to a *fixed* element of surface, a quantity less intimately connected with the condition of the matter present than is the case with the flux referred to an element of surface *moving* with the matter. Consequent upon this choice is the fact that the rôle played by radiation pressure in the final equation is not clear. Purpose of this note is to elucidate this problem.

1. Generally speaking a quantity defining the field of radiation referred to fixed elements of surface or volume differs from the same quantity referred to moving elements to an amount of the order of V/c.

For moderate velocities this difference is irrelevant, unless the coefficient of V/c is very large compared with the quantity to be computed. This case presents itself in astrophysical application of the flux of radiation $\mathbf{F}$ as is shown by the actual computation of the V/c effect.

Consider a system of rectangular axes ξ, η, ζ and matter moving with respect to this system with velocity V in the direction of $+\zeta$. Denote the quantities referred to the fixed axes by asterisks. Then if θ is the angle between the incident radiation of frequency ν, I_ν, and the axis of ζ, we have the relations:

$$\theta - \theta^* = \frac{V}{c} \sin\theta$$

$$\nu - \nu^* = -\nu \frac{V}{c} \cos\theta$$

$$I_\nu \, d\omega = I^*_{\nu^*} \, d\omega^* \left(1 - \frac{V}{c}\cos\theta\right);$$

$d\omega$ is the element of solid angle. If φ is the azimuth corresponding to the polar distance θ, we have:

$$(F^*_{\nu^*})_\xi = \int I_\nu \, d\omega \left(1 + \frac{V}{c}\cos\theta\right) [\sin\theta + (\theta^* - \theta)\cos\theta] \cos\varphi = \int I_\nu \sin\theta \cos\varphi \, d\omega$$

$$(F^*_{\nu^*})_\eta = \int I_\nu \sin\theta \sin\varphi \, d\omega$$

$$(F^*_{\nu^*})_\zeta = \int I_\nu \, d\omega \left(1 + \frac{V}{c}\cos\theta\right) [\cos\theta - (\theta^* - \theta)\sin\theta] = \int I_\nu \cos\theta \, d\omega + \frac{V}{c}\int I_\nu \, d\omega.$$

The integrations in the right-hand members involve θ, as ν^* is to be kept fixed. Hence introducing

$$I_\nu = I_{\nu*} - \nu \frac{\partial I_\nu}{\partial \nu} \frac{V}{c} \cos\theta$$

we arrive at the system of equations:

$$(F^*_{\nu^*})_\xi = (F_{\nu*})_\xi - \frac{V}{c} \nu \frac{\partial}{\partial \nu} \int I_\nu \sin\theta \cos\theta \cos\varphi \, d\omega$$

$$(F^*_{\nu^*})_\eta = (F_{\nu*})_\eta - \frac{V}{c} \nu \frac{\partial}{\partial \nu} \int I_\nu \sin\theta \cos\theta \sin\varphi \, d\omega$$

$$(F^*_{\nu^*})_\zeta = (F_{\nu*})_\zeta - \frac{V}{c} \nu \frac{\partial}{\partial \nu} \int I_\nu \cos^2\theta \, d\omega + \frac{V}{c}\int I_\nu \, d\omega.$$

Introducing the tensor with the components

$$(\Pi_\nu)_{\xi\eta} = \frac{1}{c}\int I_\nu \cos(I_\nu, \xi) \cos(I_\nu, \eta) \, d\omega,$$

[1]) S. ROSSELAND: *Astroph. Journal* 63, p. 342 etc.

integrating over all frequencies and denoting $\int \mathbf{F}_\nu \, d\nu$ by $\mathbf{F}$, etc., we arrive at the relations:

$$F_\xi^* = F_\xi + V\Pi_{\xi\zeta}$$
$$F_\eta^* = F_\eta + V\Pi_{\eta\zeta}$$
$$F_\zeta^* = F_\zeta + V\Pi_{\zeta\zeta} + \frac{V}{c}\int I \, d\omega.$$

The system of coordinates ξ, η, ζ has a special orientation with regard to the moving matter: the material velocity is parallel to the axis of ζ. Introducing a system of reference x, y, z arbitrarily orientated, the preceding relations transform into the set:

$$F_x^* = F_x + (V_x \Pi_{xx} + V_y \Pi_{yx} + V_z \Pi_{zx}) + \frac{V_x}{c}\int I \, d\omega$$
$$F_y^* = F_y + (V_x \Pi_{xy} + V_y \Pi_{yy} + V_z \Pi_{zy}) + \frac{V_y}{c}\int I \, d\omega$$
$$F_z^* = F_z + (V_x \Pi_{xz} + V_y \Pi_{yz} + V_z \Pi_{zz}) + \frac{V_z}{c}\int I \, d\omega.$$

Compare the three terms in the right hand members with the constituents of the flow of material energy. This flow consists of four terms:

the flow of kinetic energy of mass motion;

the conductive flow of thermal energy;

the vector with components $V_x p_{xx} + V_y p_{yx} + V_z p_{zx}$, etc.;

the convective flow of thermal energy;

p_{xx} etc. denotes the transport per unit time and surface element of x-momentum in the direction of the axis of x, etc.

The correspondence of the three last terms with those constituting the radiative flow $\mathbf{F}^*$ is evident.

2. The usual form of the energy equation is obtained by making up the balance between gain and loss of energy and the work done by external forces, afterwards eliminating the work done by these forces by means of the hydrodynamical equations.

The forces exerted by the radiation field on the material particles consist in the gain of momentum per unit time caused by absorption and emission. Hence if E_ν is the coefficient of emission of frequency ν, the radiation force per unit volume has the x-component:

$$\frac{1}{c}\iint \varkappa_\nu \rho I_\nu \cos(I_\nu, x) \, d\omega \, d\nu - \frac{1}{c}\iint E_\nu \cos(E_\nu, x) \, d\omega \, d\nu;$$

ρ is the density of matter, $\varkappa_\nu$ the mass-absorption coefficient. As this quantity [1]) equals:

$$-\frac{1}{c^2}\frac{\partial F_x}{\partial t} - \left[\frac{\partial \Pi_{xx}}{\partial x} + \frac{\partial \Pi_{yx}}{\partial y} + \frac{\partial \Pi_{zx}}{\partial z}\right],$$

the correspondence between the material tensor $p_{xx}, \ldots p_{zz}$ and the radiation-tensor $\Pi_{xx}, \ldots \Pi_{zz}$ is complete, apart from the term $\frac{1}{c^2}\frac{\partial \mathbf{F}}{\partial t}$.

This term however is generally unimportant.

So, if the flux of radiation is referred to the matter in motion, the equation of conservation of energy, as derived by ROSSELAND, must be supplemented by adding the radiation pressure to the material pressure and the energy of radiation to the internal energy of the matter. Then, if we introduce the operation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}$$

the energy equation takes the form:

$$\frac{d(U_r + U_i)}{dt} - \frac{p + \Pi}{\rho^2}\frac{d\rho}{dt} = -\frac{1}{\rho}\operatorname{div}(\mathbf{F}_r + \mathbf{F}_c);$$

U_r and U_i denote the radiative and internal energy per unit mass, $\mathbf{F}_r$ and $\mathbf{F}_c$ the radiative and conductive flow with reference to the matter in motion; the tensors p_{xx} etc. and Π_{xx} etc. have been supposed to reduce to the scalars p and Π. This is the equation in the form used by EDDINGTON.

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[1]) ROSSELAND, *l. c.*