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MANY-SPHERE HYDRODYNAMIC INTERACTIONS

III. THE INFLUENCE OF A PLANE WALL

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A previously developed scheme – to evaluate the (translational and rotational) mobility tensors for an arbitrary number of spheres in an unbounded fluid – is extended to include the presence of a plane wall General expressions for the friction tensors and the fluid velocity field are also obtained

1. Introduction

The hydrodynamic interactions between spherical particles in a viscous fluid play an essential role in the theory of suspensions[1]) Characteristic of these interactions via the fluid is their very long range As a consequence, the influence of boundary walls on properties of suspensions can be of importance even in cases where the vessel containing the suspension is very large The velocity of sedimentation, for example, becomes infinite in an unbounded suspension – a paradoxical situation (noticed by Smoluchowski[2])) which can be resolved[26]) by accounting for the presence of the wall supporting the fluid The effect of boundary walls on Brownian motion has been studied experimentally by means of light-scattering in a thin film cell[3]) Such wall effects may also play an important role in recent experiments on two-dimensional ordering of colloidal suspensions in this geometry[4])

A second consequence of the long range of hydrodynamic interactions is the importance of non-additivity that two-sphere hydrodynamic interactions do not suffice to describe diffusion in a suspension which is not dilute has been demonstrated both theoretically[5]) and experimentally[6]) Recently a scheme has been developed to *resum* the hydrodynamic interactions of clusters of 2, 3, 4, 5, spheres, and applied to a calculation of diffusion coefficient[7]) and effective viscosity[8]), valid up to high concentrations

The application of resummation techniques in calculating transport properties of concentrated suspensions has been made possible by the use of general

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expressions for many-sphere mobilities in an unbounded fluid, derived in refs. 9 and 10*). (The latter paper will hereafter be referred to as I.) In part II[12]) of this series the analysis given in I (performed in the static case) has been extended to the case of finite frequencies (see in this connection also ref. 13). It is the purpose of the present paper to extend the analysis of I to the case of a fluid bounded by a plane wall on which the fluid obeys a stick boundary condition†.

The influence of a plane wall on the motion of one single sphere has been studied extensively, cf. refs. 1, 14, 15 (and refs. therein); for two spheres only partial results, for special configurations, are known[16,17]). An important role in these analyses has been played by the work of Lorentz[18,19]) who obtained a solution to the following problem: given a velocity field $\boldsymbol{v}(\boldsymbol{r})$ which is a solution of Stokes' equation, find a second solution $\boldsymbol{v}'(\boldsymbol{r})$ which on the plane $z=0$ satisfies: $v'_x = -v_x$, $v'_y = -v_y$, $v'_z = v_z$. As we shall see, this result is essential to our analysis for many spheres as well.

In section 2 we formulate the problem of N spheres and a wall, within the context of the method of induced forces[9,20]). Using Lorentz' result one can formally solve the problem in terms of force-densities induced on the surfaces of the N spheres. In section 3 the moments in a multipole expansion of these induced forces are determined along the lines of paper I. General expressions for the (translational and rotational) mobility tensors of the spheres are then obtained in sections 4 and 5 as an expansion in the two parameters a/R and $a/(R^2+4l^2)^{1/2}$. Here a is a typical sphere radius and R and l are the typical distances between two spheres and between a sphere and the wall, respectively. (The latter parameter may also refer to a single sphere, in which case it equals $\frac{1}{2}a/l$.) These expressions are extensions of those given in I for the case of an unbounded fluid. Similar formulae can be obtained for the friction tensors (cf. appendix B) and the fluid velocity field (cf. eq. (4.22) and ref. 5).

Explicit expressions for the mobility tensors to third order are given in section 6; to this order the hydrodynamic interactions between at most two spheres and the wall contribute. Should the need arise to obtain results valid up to higher order in the expansion parameters, then such extensions can be obtained in a straightforward way by evaluating contractions of tensors whose expressions are given in this paper. All results and notations relevant to the reader not interested in the derivation or such extensions are summarized in section 6.

* A formal treatment of many-sphere hydrodynamic interactions has been given by Yoshizaki and Yamakawa[11]).

† If the wall is replaced by a plane free surface (on which a perfect slip boundary condition holds), the results of I may be applied at once: it is sufficient to consider the influence of image spheres, reflected with respect to the surface (cf. ref. 1).

2. Formulation of the problem using induced forces

We consider N macroscopic spheres of radii a_j ($j = 1, 2, \ldots, N$) immersed in an incompressible fluid with viscosity η. Contrary to the case considered in paper I, we will include in our analysis the presence of a single infinite plane wall. We shall represent the position of this wall by $\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = 0$, where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the wall. The centers of the spheres have positions $\boldsymbol{R}_j$ and lie in the halfspace $\boldsymbol{r} \cdot \hat{\boldsymbol{n}} > 0$.

The motion of the fluid in the halfspace $\boldsymbol{r} \cdot \hat{\boldsymbol{n}} > 0$ obeys the quasistatic Stokes equation which – within the context of the induced force method[9,20]) – reads

$$\left.\begin{aligned} &\nabla p(\boldsymbol{r}) - \eta \Delta \boldsymbol{v}(\boldsymbol{r}) = \sum_{j=1}^{N} \boldsymbol{F}_j(\boldsymbol{r}) \\ &\nabla \cdot \boldsymbol{v}(\boldsymbol{r}) = 0 \end{aligned}\right\} \quad \text{for } \boldsymbol{r} \cdot \hat{\boldsymbol{n}} > 0 . \tag{2.1}$$

Here $\boldsymbol{v}(\boldsymbol{r})$ is the velocity field and $p(\boldsymbol{r})$ the hydrostatic pressure. The induced force densities $\boldsymbol{F}_j(\boldsymbol{r})$ ($j = 1, 2, \ldots, N$) are to be chosen in such a way that

$$\boldsymbol{F}_j(\boldsymbol{r}) = 0 \qquad \text{for } |\boldsymbol{r} - \boldsymbol{R}_j| > a_j , \tag{2.2}$$

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{u}_j + \boldsymbol{\omega}_j \wedge (\boldsymbol{r} - \boldsymbol{R}_j) \qquad \text{for } |\boldsymbol{r} - \boldsymbol{R}_j| \leq a_j , \tag{2.3}$$

$$p(\boldsymbol{r}) = 0 \qquad \text{for } |\boldsymbol{r} - \boldsymbol{R}_j| < a_j , \tag{2.4}$$

so that eq. (2.1) reduces to the homogeneous Stokes equation within the fluid, supplemented by stick boundary conditions on the surfaces of the spheres. In eq. (2.3) $\boldsymbol{u}_j$ and $\boldsymbol{\omega}_j$ are the velocity and angular velocity of sphere j, respectively. On the fixed wall we also prescribe stick boundary conditions*

$$\boldsymbol{v}(\boldsymbol{r}) = 0 \quad \text{for } \boldsymbol{r} \cdot \hat{\boldsymbol{n}} = 0 . \tag{2.5}$$

It follows from the above equations that the induced forces are non-zero on the surfaces of the spheres only and are of the form

$$\boldsymbol{F}_j(\boldsymbol{r}) = a_j^{-2} \boldsymbol{f}_j(\hat{\boldsymbol{n}}_j) \delta(|\boldsymbol{r} - \boldsymbol{R}_j| - a_j) , \tag{2.6}$$

with $\hat{\boldsymbol{n}}_j$ a unit vector normal to the surface of sphere j and pointing outwards

* We remark that boundary conditions (2 3) and (2 5) uniquely determine the pressure on the surfaces of the spheres and on the wall

(cf. ref. 20). In the remainder of this section we shall construct the solution $\boldsymbol{v}(\boldsymbol{r})$, $p(\boldsymbol{r})$ of eq. (2.1) and of boundary condition (2.5) on the wall, in terms of these – as yet undetermined – induced forces.

We first note that the solution $\boldsymbol{v}_1(\boldsymbol{r})$, $p_1(\boldsymbol{r})$ of eq. (2.1) *for all* $\boldsymbol{r}$ satisfies the homogeneous Stokes equation for $\boldsymbol{r}\cdot\hat{\boldsymbol{n}}<0$ by virtue of the fact that the induced forces are zero outside the spheres and the fact that all the spheres lie in the halfspace $\boldsymbol{r}\cdot\hat{\boldsymbol{n}}>0$. We now pose the following problem

$$\left.\begin{aligned}&\nabla p(\boldsymbol{r})-\eta\Delta\boldsymbol{v}(\boldsymbol{r})=0\\&\nabla\cdot\boldsymbol{v}(\boldsymbol{r})=0\end{aligned}\right\}\quad\text{for } \boldsymbol{r}\cdot\hat{\boldsymbol{n}}<0\,, \tag{2.7}$$

$$\boldsymbol{v}(\boldsymbol{r})=-\boldsymbol{S}\cdot\boldsymbol{v}_1(\boldsymbol{r})\qquad\text{for } \boldsymbol{r}\cdot\hat{\boldsymbol{n}}=0\,, \tag{2.8}$$

where $\boldsymbol{S}\cdot\boldsymbol{v}_1(\boldsymbol{r})$ is the reflection with respect to the wall of the vectorfield $\boldsymbol{v}_1(\boldsymbol{r})$ defined above, that is to say

$$\boldsymbol{S}\equiv\boldsymbol{1}-2\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\,. \tag{2.9}$$

Here $\boldsymbol{1}$ is the unit tensor. The solution $\boldsymbol{v}_2(\boldsymbol{r})$, $p_2(\boldsymbol{r})$ for $\boldsymbol{r}\cdot\hat{\boldsymbol{n}}\leqslant 0$ of problem (2.7)–(2.8) is given by Lorentz[18]),

$$\boldsymbol{v}_2(\boldsymbol{r})=-\boldsymbol{S}\cdot\boldsymbol{v}_1(\boldsymbol{r})-2(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})\nabla\boldsymbol{v}_1(\boldsymbol{r})\cdot\hat{\boldsymbol{n}}+\eta^{-1}(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})^2\nabla p_1(\boldsymbol{r})\,, \tag{2.10}$$

$$p_2(\boldsymbol{r})=p_1(\boldsymbol{r})+2(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}\cdot\nabla p_1(\boldsymbol{r})-4\eta\hat{\boldsymbol{n}}\cdot\nabla\boldsymbol{v}_1(\boldsymbol{r})\cdot\hat{\boldsymbol{n}}\,. \tag{2.11}$$

That $\boldsymbol{v}_2(\boldsymbol{r})$ given by eq. (2.10) satisfies (2.8) is obvious; using the fact that $\boldsymbol{v}_1(\boldsymbol{r})$, $p_1(\boldsymbol{r})$ satisfy eq. (2.7), one may verify by substitution that $\boldsymbol{v}_2(\boldsymbol{r})$, $p_2(\boldsymbol{r})$ are also a solution of eq. (2.7).

It is not difficult to see that $\boldsymbol{v}_3(\boldsymbol{r})$, $p_3(\boldsymbol{r})$, given by

$$\boldsymbol{v}_3(\boldsymbol{r})=\boldsymbol{S}\cdot\boldsymbol{v}_2(\boldsymbol{S}\cdot\boldsymbol{r})\,,\quad p_3(\boldsymbol{r})=p_2(\boldsymbol{S}\cdot\boldsymbol{r})\,, \tag{2.12}$$

are for $\boldsymbol{r}\cdot\hat{\boldsymbol{n}}>0$ a solution of the homogeneous Stokes equation, with $\boldsymbol{v}_3(\boldsymbol{r})=-\boldsymbol{v}_1(\boldsymbol{r})$ on the wall. For every set of induced forces the solution $\boldsymbol{v}(\boldsymbol{r})$, $p(\boldsymbol{r})$ of (2.1) and (2.5) is therefore given by the sum

$$\boldsymbol{v}(\boldsymbol{r})=\boldsymbol{v}_1(\boldsymbol{r})+\boldsymbol{v}_3(\boldsymbol{r})\,,\quad p(\boldsymbol{r})=p_1(\boldsymbol{r})+p_3(\boldsymbol{r})\,. \tag{2.13}$$

In order to give an expression for $\boldsymbol{v}(\boldsymbol{r})$ and $p(\boldsymbol{r})$ in terms of the induced forces, it is convenient to introduce Fourier transforms of fields defined for all

$\boldsymbol{r}$, as for the velocity field

$$\boldsymbol{v}_1(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{v}_1(\boldsymbol{r}) \tag{2.14}$$

and similarly for the pressure field. The Fourier transform of an induced force density $\boldsymbol{F}_j(\boldsymbol{r})$ is defined in a reference frame in which sphere j is at the origin

$$\boldsymbol{F}_j(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_j)} \boldsymbol{F}_j(\boldsymbol{r}) \,. \tag{2.15}$$

In wavevector representation one then has

$$\mathrm{i}\boldsymbol{k} p_1(\boldsymbol{k}) + \eta k^2 \boldsymbol{v}_1(\boldsymbol{k}) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} \boldsymbol{F}_j(\boldsymbol{k}) \,, \tag{2.16}$$

$$\boldsymbol{k} \cdot \boldsymbol{v}_1(\boldsymbol{k}) = 0 \,, \tag{2.17}$$

with $k \equiv |\boldsymbol{k}|$. If one contracts both sides of eq. (2.16) with the tensor $\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$ (where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$ is the unit vector in the direction of $\boldsymbol{k}$) one obtains with eq. (2.17)

$$\eta k^2 \boldsymbol{v}_1(\boldsymbol{k}) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} (\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) \cdot \boldsymbol{F}_j(\boldsymbol{k}) \,. \tag{2.18}$$

Similarly, a contraction of eq. (2.16) with $\hat{\boldsymbol{k}}$ gives

$$\mathrm{i}k p_1(\boldsymbol{k}) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} \hat{\boldsymbol{k}} \cdot \boldsymbol{F}_j(\boldsymbol{k}) \,. \tag{2.19}$$

Eqs. (2.16) and (2.17) therefore have a solution

$$\boldsymbol{v}_1(\boldsymbol{k}) = \eta^{-1} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} k^{-2} (\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) \cdot \boldsymbol{F}_j(\boldsymbol{k}) \,, \tag{2.20}$$

$$p_1(\boldsymbol{k}) = -\mathrm{i} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} k^{-1} \hat{\boldsymbol{k}} \cdot \boldsymbol{F}_j(\boldsymbol{k}) \,. \tag{2.21}$$

According to eq. (2.10) one has for $\boldsymbol{v}_2(\boldsymbol{k})$

$$v_2(k) = -\mathbf{S}\cdot v_1(k) + 2\hat{n}\cdot\frac{\partial}{\partial k}[\hat{n}\cdot v_1(k)k] - \mathrm{i}\eta^{-1}\hat{n}\cdot\frac{\partial}{\partial k}\,\hat{n}\cdot\frac{\partial}{\partial k}[kp_1(k)]\,. \tag{2.22}$$

Substitution of eqs. (2.20) and (2.21) into eq. (2.22) yields an expression for $v_2(k)$ in terms of the induced forces.

From eq. (2.12) one readily finds

$$v_3(k) = \mathbf{S}\cdot v_2(\mathbf{S}\cdot k)\,. \tag{2.23}$$

The solution $v(r)$ of eq. (2.1) with boundary condition (2.5) on the wall – for given induced forces – is therefore, according to eq. (2.13), given by

$$v(r) = v_0(r) + (2\pi)^{-3}\int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k\cdot r}[v_1(k) + v_3(k)]\,, \tag{2.24}$$

with

$$\begin{aligned} v_1(k) + v_3(k) = \eta^{-1}\sum_{j=1}^{N}\Big\{ & \mathrm{e}^{-\mathrm{i}k\cdot R_j}k^{-2}(\mathbf{1} - \hat{k}\hat{k})\cdot F_j(k) \\ & - \mathrm{e}^{-\mathrm{i}k\cdot \mathbf{S}\cdot R_j}k^{-2}(\mathbf{1} - (\mathbf{S}\cdot\hat{k})(\mathbf{S}\cdot\hat{k}))\cdot F_j(\mathbf{S}\cdot k) \\ & - 2\hat{n}\cdot\frac{\partial}{\partial k}[\mathrm{e}^{-\mathrm{i}k\cdot\mathbf{S}\cdot R_j}k^{-1}\hat{k}\hat{n}\cdot(\mathbf{1} - (\mathbf{S}\cdot\hat{k})(\mathbf{S}\cdot\hat{k}))\cdot F_j(\mathbf{S}\cdot k)] \\ & - \hat{n}\cdot\frac{\partial}{\partial k}\,\hat{n}\cdot\frac{\partial}{\partial k}[\mathrm{e}^{-\mathrm{i}k\cdot\mathbf{S}\cdot R_j}\hat{k}\hat{k}\cdot\mathbf{S}\cdot F_j(\mathbf{S}\cdot k)]\Big\}\,. \end{aligned} \tag{2.25}$$

In eq. (2.24), $v_0(r)$ is a solution of (2.1) and (2.5) in the absence of induced forces on the spheres and is therefore the velocity field unperturbed by the presence of the N spheres. For convenience we shall assume that the unperturbed fluid is at rest,

$$v_0(r) = 0 \tag{2.26}$$

(see however in this connection the concluding remarks of paper I).

We shall now proceed to determine the induced force density on the surfaces of the spheres which follows from the requirement that $v(r)$ satisfies stick boundary conditions on these surfaces. Note that for this purpose knowledge of the pressure field $p(r)$ is not required.

3. Determination of the induced forces

To determine the induced forces on the surfaces of the spheres we shall use the general scheme developed in paper I. By analyzing the velocity surface moments

$$\overline{\overline{\hat{n}_j^p} v(r)}^{S_j} \equiv (4\pi a_j^2)^{-1} a_j^{-p} \int \mathrm{d}r\, \overline{(r - R_j)^p}\, v(r) \delta(|r - R_j| - a_j)\,, \tag{3.1}$$

one obtains a hierarchy of equations for the irreducible force multipoles

$$F_j^{(p+1)} \equiv a_j^{-p} (p!)^{-1} \int \mathrm{d}r\, \overline{(r - R_j)^p}\, F_j(r) = (p!)^{-1} \int \mathrm{d}\hat{n}_j\, \overline{\hat{n}_j^p}\, f_j(\hat{n}_j)\,. \tag{3.2}$$

In the above equations the notation $\overline{b^p}$ denotes an irreducible tensor of rank p (i.e. a tensor traceless and symmetric in any pair of its indices) constructed from the p-fold ordered product of the vector b*. We shall give an outline of this procedure below. For a more elaborate exposition one is referred to paper I.

Since $v(r)$ should satisfy eq. (2.3) on the surface S_j of sphere j, one finds for the velocity surface moments (3.1) the set of equations

$$\overline{v(r)}^{S_j} = u_j\,, \quad \overline{\hat{n}_j v(r)}^{S_j} = \tfrac{1}{3} a_j \epsilon \cdot \omega_j\,, \quad \overline{\overline{\hat{n}_j^p}\, v(r)}^{S_j} = 0 \quad (p \geqslant 2)\,. \tag{3.3}$$

Here ϵ is the Levi–Civita tensor.

The induced force may be written in terms of the irreducible force multipoles (3.2), by means of the expansion (paper I, appendix A)

$$F_j(k) = \sum_{p=0}^{\infty} (2p+1)!!\, \mathrm{i}^p \left(\overline{\frac{\partial^p}{\partial (a_j k)^p}} \frac{\sin a_j k}{a_j k} \right) \odot F_j^{(p+1)}\,. \tag{3.4}$$

Here $(2p+1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2p-1) \cdot (2p+1)$ and the dot $\odot$ denotes a full contraction of the first p indices of $F^{(p+1)}$ (which is a tensor of rank $p+1$) with the p indices of the tensor between brackets. The surface moments (3.1) may also be written as

* For $p = 1, 2, 3$ one has e.g. (cf. ref. 21)

$$\overline{b_\mu} = b_\mu,\ \overline{b_\mu b_\nu} = b_\mu b_\nu - \tfrac{1}{3} b^2 \delta_{\mu\nu},\ \overline{b_\mu b_\nu b_\lambda} = b_\mu b_\nu b_\lambda - \tfrac{1}{5} b^2 (\delta_{\mu\nu} b_\lambda + \delta_{\mu\lambda} b_\nu + \delta_{\nu\lambda} b_\mu)\,,$$

where $b \equiv |b|$ and μ, ν, λ denote cartesian components.

$$\overline{\overline{\hat{\boldsymbol{n}}_j^p}\,\boldsymbol{v}(\boldsymbol{r})}^{S_j} = (-\mathrm{i})^p(2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}\left(\frac{\overline{\partial^p}}{\partial(a_j\boldsymbol{k})^p}\,\frac{\sin a_j k}{a_j k}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j}\boldsymbol{v}(\boldsymbol{k})\,. \tag{3.5}$$

We notice here the useful identity[22])

$$\frac{\overline{\partial^p}}{\partial\boldsymbol{k}^p}\,\frac{\sin k}{k} = \overline{\hat{\boldsymbol{k}}^p}(-1)^p j_p(k)\,, \tag{3.6}$$

with j_p a spherical Bessel function*.

If one evaluates the surface moments (3.5), using eqs. (2.24)–(2.26) and (3.4) and equates these moments to the values given in eq. (3.3), one obtains the following hierarchy of equations ($i = 1, 2, \ldots, N$)

$$\begin{aligned}
6\pi\eta a_i \boldsymbol{u}_i &= \sum_{j=1}^{N}\sum_{m=1}^{\infty}\left(\boldsymbol{A}_{ij}^{(1,m)} + \boldsymbol{W}_{ij}^{(1,m)}\right)\odot \boldsymbol{F}_j^{(m)}\,,\\
6\pi\eta a_i^2 \boldsymbol{\epsilon}\cdot\boldsymbol{\omega}_i &= \sum_{j=1}^{N}\sum_{m=1}^{\infty}\left(\boldsymbol{A}_{ij}^{(2\mathrm{a},m)} + \boldsymbol{W}_{ij}^{(2\mathrm{a},m)}\right)\odot \boldsymbol{F}_j^{(m)}\,,\\
0 &= \sum_{j=1}^{N}\sum_{m=1}^{\infty}\left(\boldsymbol{A}_{ij}^{(n,m)} + \boldsymbol{W}_{ij}^{(n,m)}\right)\odot \boldsymbol{F}_j^{(m)} \quad (n = 2\mathrm{s}, 3, 4, \ldots)\,.
\end{aligned} \tag{3.7}$$

The so-called connectors $\boldsymbol{A}_{ij}^{(n,m)}$ and $\boldsymbol{W}_{ij}^{(n,m)}$ are defined by

$$\begin{aligned}
\boldsymbol{A}_{ij}^{(n,m)} &= \tfrac{3}{4}\pi^{-2} a_i(-\mathrm{i})^{m-n}(2n-1)!!(2m-1)!!\int \mathrm{d}\boldsymbol{k}\;\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{R}_j-\boldsymbol{R}_i)}\\
&\quad\times k^{-2} j_{n-1}(a_i k) j_{m-1}(a_j k)\,\overline{\hat{\boldsymbol{k}}^{n-1}}(\boldsymbol{1}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\,\overline{\hat{\boldsymbol{k}}^{m-1}}\,,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\boldsymbol{W}_{ij}^{(n,m)} &= -\tfrac{3}{4}\pi^{-2} a_i(-\mathrm{i})^{m-n}(2n-1)!!(2m-1)!!\int \mathrm{d}\boldsymbol{k}\;\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_i}\\
&\quad\times j_{n-1}(a_i k)\,\overline{\hat{\boldsymbol{k}}^{n-1}}\Big\{(\boldsymbol{1}-(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})(\boldsymbol{S}\cdot\hat{\boldsymbol{k}}))\,\overline{(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})^{m-1}}\,\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{k}}\cdot\boldsymbol{S}\cdot\boldsymbol{R}_j}k^{-2}j_{m-1}(a_j k)\\
&\quad+2\hat{\boldsymbol{n}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\left[\hat{\boldsymbol{k}}\hat{\boldsymbol{n}}\cdot(\boldsymbol{1}-(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})(\boldsymbol{S}\cdot\hat{\boldsymbol{k}}))\,\overline{(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})^{m-1}}\,\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{k}}\cdot\boldsymbol{S}\cdot\boldsymbol{R}_j}k^{-1}j_{m-1}(a_j k)\right]\\
&\quad+\hat{\boldsymbol{n}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\,\hat{\boldsymbol{n}}\cdot\frac{\partial}{\partial\boldsymbol{k}}\left[\hat{\boldsymbol{k}}(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})\,\overline{(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})^{m-1}}\,\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{k}}\cdot\boldsymbol{S}\cdot\boldsymbol{R}_j}j_{m-1}(a_j k)\right]\Big\}\,,
\end{aligned} \tag{3.9}$$

* The function $j_p(x)$ is related to the Bessel function of order $p+\frac{1}{2}$ by $j_p(x) = (\frac{1}{2}\pi/x)^{1/2}J_{p+1/2}(x)$.

where we have also made the substitution (3.6). The dot $\odot$ in e.g. $\boldsymbol{A}_{ij}^{(n,m)} \odot \boldsymbol{F}_j^{(m)}$ prescribes an m-fold contraction with the (nesting) convention that the last index of the first tensor is contracted with the first index of the second tensor, etc. For example

$$(\boldsymbol{A}^{(1,3)} \odot \boldsymbol{F}^{(3)})_\alpha = \sum_{\beta\gamma\delta} A^{(1,3)}_{\alpha,\beta\gamma\delta} F^{(3)}_{\delta\gamma\beta} \, . \tag{3.10}$$

The connectors $\boldsymbol{A}_{ij}^{(n,m)}$ and $\boldsymbol{W}_{ij}^{(n,m)}$ defined above are tensors of rank $n+m$ which are irreducible in their first $n-1$ and last $m-1$ indices. The connectors $\boldsymbol{A}$ do not depend on the position of the wall and were given in paper I where the case of an unbounded fluid was considered. The stick boundary condition (2.5) on the wall is accounted for by the connectors $\boldsymbol{W}$. In eq. (3.7) we have decomposed these connectors for $n = 2$ as follows

$$\boldsymbol{A}_{ij}^{(2,m)} = \boldsymbol{A}_{ij}^{(2\mathrm{a},m)} + \boldsymbol{A}_{ij}^{(2\mathrm{s},m)}, \quad \boldsymbol{W}_{ij}^{(2,m)} = \boldsymbol{W}_{ij}^{(2\mathrm{a},m)} + \boldsymbol{W}_{ij}^{(2\mathrm{s},m)}, \tag{3.11}$$

where $\boldsymbol{A}^{(2\mathrm{a},m)}$, $\boldsymbol{W}^{(2\mathrm{a},m)}$ are antisymmetric and $\boldsymbol{A}^{(2\mathrm{s}\,m)}$, $\boldsymbol{W}^{(2\mathrm{s},m)}$ are symmetric in the first two indices. We remark that both $\boldsymbol{A}^{(2,m)}$ and $\boldsymbol{W}^{(2,m)}$ are traceless in the first two indices. This property follows from the fact (cf. paper I) that in view of eq. (2.1)

$$\overline{\hat{\boldsymbol{n}}_j \cdot \boldsymbol{v}(\boldsymbol{r})}^{S_j} = (4\pi a_j^2)^{-1} \int_{|\boldsymbol{r}-\boldsymbol{R}_j|\leq a_j} \mathrm{d}\boldsymbol{r} \boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{r}) = 0 \, , \tag{3.12}$$

irrespective of the boundary conditions.

As in paper I, it is convenient to separate the set of connectors $\boldsymbol{A}_{ii}^{(n,m)}$ – which do not depend on the positions of the spheres – from the connectors $\boldsymbol{A}_{ij}^{(n,m)}$ with $i \neq j$. We therefore define

$$\boldsymbol{B}^{(n,m)} = -\boldsymbol{A}_{ii}^{(n,m)}, \quad \boldsymbol{C}_{ij}^{(n,m)} = \boldsymbol{A}_{ij}^{(n,m)}(1-\delta_{ij}) + \boldsymbol{W}_{ij}^{(n,m)} \, . \tag{3.13}$$

According to eq. (3.8), $\boldsymbol{B}^{(n,m)}$ is given by

$$\boldsymbol{B}^{(n,m)} = -\tfrac{3}{4}\pi^{-2}(2n-1)!!(2m-1)!!(-\mathrm{i})^{m-n} \int \mathrm{d}\hat{\boldsymbol{k}}\, \overline{\hat{\boldsymbol{k}}^{n-1}}(\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\, \overline{\hat{\boldsymbol{k}}^{m-1}}$$
$$\times \int_0^\infty \mathrm{d}k j_{n-1}(k) j_{m-1}(k) \, . \tag{3.14}$$

Obviously, $\boldsymbol{B}^{(n,m)} = 0$ if n is even and m odd, or the other way around, since in that case the first integrand is an odd function of $\hat{k}$. On the other hand, if both n and m are even or odd, the k-integration gives zero for $n \neq m$, by virtue of an orthogonality property of Bessel functions[23]). One therefore has

$$\boldsymbol{B}^{(n,m)} = 0 \quad \text{if } n \neq m\,, \tag{3.15}$$

as asserted in paper I.

4. Sphere mobilities and fluid velocity field

The force $\boldsymbol{K}_j$ and torque $\boldsymbol{T}_j$ exerted by the fluid on sphere j are given by

$$\boldsymbol{K}_j = -\int_{S_j} \mathrm{d}S \boldsymbol{P}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}}_j\,, \quad \boldsymbol{T}_j = -\int_{S_j} \mathrm{d}S (\boldsymbol{r} - \boldsymbol{R}_j) \wedge \boldsymbol{P}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}}_j\,, \tag{4.1}$$

where the pressure tensor $\boldsymbol{P}$ has components

$$P_{\mu\nu} = p\delta_{\mu\nu} - \eta\left(\frac{\partial v_\nu}{\partial r_\mu} + \frac{\partial v_\mu}{\partial r_\nu}\right). \tag{4.2}$$

In terms of multipoles of the induced force one has (cf. eqs. (I-3.10)–(I-3.12))

$$\boldsymbol{F}_j^{(1)} = -\boldsymbol{K}_j\,, \quad \boldsymbol{F}_j^{(2a)} = -(2a_j)^{-1} \boldsymbol{\epsilon} \cdot \boldsymbol{T}_j\,, \tag{4.3}$$

where $\boldsymbol{F}_j^{(2a)}$ is the antisymmetric part of $\boldsymbol{F}_j^{(2)}$.

Using eqs. (3.13), (3.15) and (4.3) one may write the hierarchy of equations (3.7) in the form ($i = 1, 2, \ldots, N$)

$$\begin{aligned}
6\pi\eta a_i \boldsymbol{u}_i &= -\boldsymbol{K}_i - \sum_{j=1}^{N} \boldsymbol{C}_{ij}^{(1,1)} \cdot \boldsymbol{K}_j - \sum_{j=1}^{N} (2a_j)^{-1} \boldsymbol{C}_{ij}^{(1,2a)} : \boldsymbol{\epsilon} \cdot \boldsymbol{T}_j \\
&\quad + \sum_{j=1}^{N} \sum_{m=2}^{\infty}{}' \boldsymbol{C}_{ij}^{(1,m)} \odot \boldsymbol{F}_j^{(m)}\,, \\
12\pi\eta a_i^2 \boldsymbol{\omega}_i &= \sum_{j=1}^{N} \boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2a,1)} \cdot \boldsymbol{K}_j - 3(2a_i)^{-1} \boldsymbol{T}_i + \sum_{j=1}^{N} (2a_j)^{-1} \boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2a,2a)} : \boldsymbol{\epsilon} \cdot \boldsymbol{T}_j \\
&\quad - \sum_{j=1}^{N} \sum_{m=2}^{\infty}{}' \boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2a,m)} \odot \boldsymbol{F}_j^{(m)}\,,
\end{aligned} \tag{4.4}$$

$$\boldsymbol{B}^{(n,m)} \odot \boldsymbol{F}_i^{(n)} = -\sum_{j=1}^{N} \boldsymbol{C}_{ij}^{(n,1)} \cdot \boldsymbol{K}_j - \sum_{j=1}^{N} (2a_j)^{-1} \boldsymbol{C}_{ij}^{(n,2a)} : \boldsymbol{\epsilon} \cdot \boldsymbol{T}_j$$
$$+ \sum_{j=1}^{N} \sum_{m=2}^{\infty}{}' \boldsymbol{C}_{ij}^{(n,m)} \odot \boldsymbol{F}_j^{(m)} \quad (n = 2s, 3, 4, \ldots) . \qquad (4.4) \text{ contd.}$$

These equations have the same form as eqs. (I-5.2)–(I-5.5) for the case of an unbounded fluid.

In the above equations use has been made of the formulae

$$\boldsymbol{B}^{(1,1)} = -\boldsymbol{1}, \quad \boldsymbol{B}^{(2,2)} = \boldsymbol{B}^{(2s,2s)} + \boldsymbol{B}^{(2a,2a)},$$
$$\boldsymbol{B}^{(2a,2a)} : \boldsymbol{F}_i^{(2)} = -\tfrac{3}{2} \boldsymbol{F}_i^{(2a)}, \quad \boldsymbol{\epsilon} : \boldsymbol{\epsilon} = -2\boldsymbol{1}, \qquad (4.5)$$

cf. paper I. Furthermore, we have denoted by $\boldsymbol{C}^{(n,2s)}$ and $\boldsymbol{C}^{(n,2a)}$ those parts of the connector $\boldsymbol{C}^{(n,2)}$ which are respectively (traceless) symmetric* and antisymmetric in the last two indices. The prime in the sum over m in eqs. (4.4) denotes a summation over all integer values $m \geqslant 2$ with the proviso that for $m = 2$ only the symmetric part of the connectors and multipoles is included in the summation, e.g.

$$\sum_{m=2}^{\infty}{}' \boldsymbol{C}_{ij}^{(n,m)} \odot \boldsymbol{F}_j^{(m)} = \boldsymbol{C}_{ij}^{(n,2s)} : \boldsymbol{F}_j^{(2s)} + \sum_{m=3}^{\infty} \boldsymbol{C}_{ij}^{(n,m)} \odot \boldsymbol{F}_j^{(m)} . \qquad (4.6)$$

Using the hierarchy of equations (4.4), one may formally eliminate $\boldsymbol{F}^{(2s)}$ and $\boldsymbol{F}^{(n)}$ ($n \geqslant 3$) in the first two equations in favor of $\boldsymbol{K}$ and $\boldsymbol{T}$. This procedure leads to linear relations of the form ($i = 1, 2, \ldots, N$)

$$\boldsymbol{u}_i = -\sum_{j=1}^{N} \boldsymbol{\mu}_{ij}^{TT} \cdot \boldsymbol{K}_j - \sum_{j=1}^{N} \boldsymbol{\mu}_{ij}^{TR} \cdot \boldsymbol{T}_j ,$$
$$\boldsymbol{\omega}_i = -\sum_{j=1}^{N} \boldsymbol{\mu}_{ij}^{RT} \cdot \boldsymbol{K}_j - \sum_{j=1}^{N} \boldsymbol{\mu}_{ij}^{RR} \cdot \boldsymbol{T}_j . \qquad (4.7)$$

Here $\boldsymbol{\mu}_{ij}^{TT}$ is the translational mobility tensor, $\boldsymbol{\mu}_{ij}^{RR}$ the rotational mobility tensor, and the tensors $\boldsymbol{\mu}_{ij}^{TR}$ and $\boldsymbol{\mu}_{ij}^{RT}$ couple translational and rotational motion. The expressions for these mobility tensors, which follow from eqs. (4.4), have the same form as in an unbounded fluid (cf. eqs. (I-5.16)–(I-5.19)†)

* The fact that $\boldsymbol{C}^{(n,2)}$ is traceless in the last two indices follows from the symmetry of the connectors discussed below, and from the fact that $\boldsymbol{C}^{(2,n)}$ is traceless in its first two indices (cf. section 3).

† There is an obvious misprint in eq. (I-5.19): for $\boldsymbol{A}_{isl}^{(m_s,1)}$ one should read $\boldsymbol{A}_{isl}^{(m_s,2a)} : \boldsymbol{\epsilon}$.

$$6\pi\eta a_i \boldsymbol{\mu}_{ij}^{\mathrm{TT}} = \boldsymbol{1}\delta_{ij} + \boldsymbol{C}_{ij}^{(1,1)} + \sum_{s=1}^{\infty} \sum_{m_1=2}^{\infty}{}' \cdots \sum_{m_s=2}^{\infty}{}' \sum_{j_1=1}^{N} \cdots \sum_{j_s=1}^{N} \boldsymbol{C}_{ij_1}^{(1,m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \ldots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,1)}, \tag{4.8}$$

$$8\pi\eta a_i^2 a_j \boldsymbol{\mu}_{ij}^{\mathrm{RR}} = \boldsymbol{1}\delta_{ij} - \tfrac{1}{3}\boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2\mathrm{a},2\mathrm{a})} : \boldsymbol{\epsilon} - \tfrac{1}{3} \sum_{s=1}^{\infty} \sum_{m_1=2}^{\infty}{}' \cdots \sum_{m_s=2}^{\infty}{}' \sum_{j_1=1}^{N} \cdots \sum_{j_s=1}^{N} \boldsymbol{\epsilon} : \boldsymbol{C}_{ij_1}^{(2\mathrm{a},m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \ldots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,2\mathrm{a})} : \boldsymbol{\epsilon}, \tag{4.9}$$

$$12\pi\eta a_i^2 \boldsymbol{\mu}_{ij}^{\mathrm{RT}} = -\boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2\mathrm{a},1)} - \sum_{s=1}^{\infty} \sum_{m_1=2}^{\infty}{}' \cdots \sum_{m_s=2}^{\infty}{}' \sum_{j_1=1}^{N} \cdots \sum_{j_s=1}^{N} \boldsymbol{\epsilon} : \boldsymbol{C}_{ij_1}^{(2\mathrm{a},m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \ldots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,1)}, \tag{4.10}$$

$$12\pi\eta a_i a_j \boldsymbol{\mu}_{ij}^{\mathrm{TR}} = \boldsymbol{C}_{ij}^{(1,2\mathrm{a})} : \boldsymbol{\epsilon} + \sum_{s=1}^{\infty} \sum_{m_1=2}^{\infty}{}' \cdots \sum_{m_s=2}^{\infty}{}' \sum_{j_1=1}^{N} \cdots \sum_{j_s=1}^{N} \boldsymbol{C}_{ij_1}^{(1,m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \ldots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,2\mathrm{a})} : \boldsymbol{\epsilon}. \tag{4.11}$$

Here $\boldsymbol{B}^{(n,n)^{-1}}$ is, for $n \geq 3$, the generalized inverse of $\boldsymbol{B}^{(n\,n)}$ in the space of tensors of rank n which are irreducible in their first $n-1$ indices. (The existence of this inverse was demonstrated explicitly in ref. 7.) For the case $n = 2s$ one has, cf. paper I,

$$\boldsymbol{C}_{ij}^{(m,2s)} : \boldsymbol{B}^{(2s,2s)^{-1}} = -\tfrac{10}{9}\boldsymbol{C}_{ij}^{(m,2s)}. \tag{4.12}$$

It follows generally from the Stokes equation with stick boundary conditions that the mobilities defined in eq. (4.7) have the properties[1])

$$\boldsymbol{\mu}_{ij}^{\mathrm{TT}} = \tilde{\boldsymbol{\mu}}_{ji}^{\mathrm{TT}}, \quad \boldsymbol{\mu}_{ij}^{\mathrm{RR}} = \tilde{\boldsymbol{\mu}}_{ji}^{\mathrm{RR}}, \quad \boldsymbol{\mu}_{ij}^{\mathrm{RT}} = \tilde{\boldsymbol{\mu}}_{ji}^{\mathrm{TR}}, \tag{4.13}$$

where $\tilde{\boldsymbol{\mu}}_{ij}$ is the transposed of $\boldsymbol{\mu}_{ij}$. Within the present scheme these symmetry relations are (as in the case of an unbounded fluid) a direct consequence of the symmetries of the connectors

$$a_j \tilde{\boldsymbol{A}}_{ij}^{(n,m)} = a_i \boldsymbol{A}_{ji}^{(m,n)}, \quad a_j \tilde{\boldsymbol{W}}_{ij}^{(n,m)} = a_i \boldsymbol{W}_{ji}^{(m,n)}. \tag{4.14}$$

Here $\tilde{\boldsymbol{T}}$ is the generalized transposed of a tensor $\boldsymbol{T}$ of arbitrary rank p,

$$(\tilde{\boldsymbol{T}})_{\mu_1\mu_2 \cdots \mu_{p-1}\mu_p} \equiv (\boldsymbol{T})_{\mu_p\mu_{p-1} \cdots \mu_2\mu_1} . \tag{4.15}$$

That the connectors $\boldsymbol{A}$ satisfy (4.14) is evident from their definition (3.8). It is possible to write also expression (3.9) for the wall connectors $\boldsymbol{W}$ in a manifestly symmetric form.

The velocity field of the fluid at point $\boldsymbol{r}$ may similarly be expressed in terms of the forces and torques exerted by the fluid on the spheres

$$\boldsymbol{v}(\boldsymbol{r}) = -\sum_{j=1}^{N} \boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{r}) \cdot \boldsymbol{K}_j - \sum_{j=1}^{N} \boldsymbol{S}_j^{\mathrm{R}}(\boldsymbol{r}) \cdot \boldsymbol{T}_j . \tag{4.16}$$

The tensors $\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{r})$, $\boldsymbol{S}_j^{\mathrm{R}}(\boldsymbol{r})$ defined above (which were not considered in paper I) can very simply be derived from the general expressions for the mobilities for $N+1$ spheres by putting $\boldsymbol{R}_{N+1} = \boldsymbol{r}$ and taking the limit $a_{N+1} \to 0$ (cf. ref. 5),

$$\left.\begin{aligned} \boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{r}) &= \lim_{a_{N+1}\to 0} \boldsymbol{\mu}_{N+1,j}^{\mathrm{TT}} \Big|_{\boldsymbol{R}_{N+1}=\boldsymbol{r}} \\ \boldsymbol{S}_j^{\mathrm{R}}(\boldsymbol{r}) &= \lim_{a_{N+1}\to 0} \boldsymbol{\mu}_{N+1,j}^{\mathrm{TR}} \Big|_{\boldsymbol{R}_{N+1}=\boldsymbol{r}} \end{aligned}\right\} \quad j = 1, 2, \ldots, N. \tag{4.17}$$

These formulae are based on the idea that the velocity field can be probed with the aid of an infinitesimally small test sphere. In view of this obvious physical interpretation, the (straightforward) formal proof of eq. (4.17) is omitted.

5. Evaluation of the connectors

General results, useful for an evaluation of the connectors $\boldsymbol{A}_{ij}^{(n,m)}$, have been given in paper I. By an extension of these arguments we shall give below explicit expressions for these connectors*, as well as for the wall connectors $\boldsymbol{W}_{ij}^{(n,m)}$. One then has explicit expressions for all the connectors appearing in formulae (4.8)–(4.11) for the mobilities.

Definition (3.8) of the connector $\boldsymbol{A}_{ij}^{(n,m)}$ may be written in the form

$$\boldsymbol{A}_{ij}^{(n,m)} = \tfrac{3}{4}\pi^{-2} a_i (-1)^{n+1} (2n-1)!!(2m-1)!! \int \mathrm{d}\boldsymbol{k} \, \frac{\overline{\partial^{n-1}}}{\partial \boldsymbol{R}_{ij}^{n-1}} (\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) \frac{\overline{\partial^{m-1}}}{\partial \boldsymbol{R}_{ij}^{m-1}} \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{R}_{ij}} \times k^{-(n+m)} j_{n-1}(a_i k) j_{m-1}(a_j k) , \tag{5.1}$$

* Similar expressions have also been obtained by Tough[24]).

where $\boldsymbol{R}_{ij} \equiv \boldsymbol{R}_j - \boldsymbol{R}_i$. Expanding the Bessel functions around $k = 0$ one may write

$$(2n-1)!!(2m-1)!!k^{-(n+m+1)} j_{n-1}(a_i k) j_{m-1}(a_j k) = a_i^{n-1} a_j^{m-1} k^{-3}\left[1 - \frac{1}{2}k^2\left(\frac{a_i^2}{2n+1} + \frac{a_j^2}{2m+1}\right)\right] + \mathscr{R}(k), \tag{5.2}$$

where $\mathscr{R}(z)$ is analytic in the complex plane and bounded for large $|z|$ by $\exp[(a_i + a_j)|z|]$.

That $\mathscr{R}(k)$ gives a vanishing contribution upon integration to $\boldsymbol{A}_{ij}$ if $i \neq j$ may be seen as follows*: upon substitution of eq. (5.2) into eq. (5.1) one finds that this contribution is of the form

$$\int \mathrm{d}\boldsymbol{k}(\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_{ij}} k\mathscr{R}(k) = 2\pi \int_{-\infty}^{\infty} \mathrm{d}k \left(\boldsymbol{1}k^2 + \frac{\partial^2}{\partial \boldsymbol{R}_{ij}^2}\right) \frac{\sin kR_{ij}}{R_{ij}} \mathscr{R}(k), \tag{5.3}$$

where $R_{ij} \equiv |\boldsymbol{R}_{ij}|$ and we have used the fact that $\mathscr{R}(-k) = \mathscr{R}(k)$. The integral in (5.3) equals zero if $i \neq j$ (in which case one necessarily has $R_{ij} > a_i + a_j$), as a consequence of the Paley–Wiener theorem (cf. e.g. ref. 25).

One therefore has the result (cf. paper I)

$$\boldsymbol{A}_{ij}^{(n,m)} = \boldsymbol{G}_{ij}^{(n,m)} + \boldsymbol{H}_{ij}^{(n,m)} \quad (i \neq j), \tag{5.4}$$

with

$$\boldsymbol{G}_{ij}^{(n,m)} = (-1)^{n+1} \frac{3}{4} a_i^n a_j^{m-1} \frac{\overrightarrow{\partial^{n-1}}}{\partial \boldsymbol{R}_{ij}^{n-1}} \frac{\boldsymbol{1} + \hat{\boldsymbol{r}}_{ij}\hat{\boldsymbol{r}}_{ij}}{R_{ij}} \frac{\overleftarrow{\partial^{m-1}}}{\partial \boldsymbol{R}_{ij}^{m-1}}, \tag{5.5}$$

$$\boldsymbol{H}_{ij}^{(n,m)} = (-1)^n \frac{3}{4} a_i^n a_j^{m-1} \left(\frac{a_i^2}{2n+1} + \frac{a_j^2}{2m+1}\right) \frac{\overrightarrow{\partial^{n-1}}}{\partial \boldsymbol{R}_{ij}^{n-1}} \frac{\partial^2}{\partial \boldsymbol{R}_{ij}^2} \frac{\overleftarrow{\partial^{m-1}}}{\partial \boldsymbol{R}_{ij}^{m-1}} R_{ij}^{-1}, \tag{5.6}$$

where use has been made of the formulae (valid for $\boldsymbol{r} \neq 0$)

$$\int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}})k^{-2} = \pi^2(\boldsymbol{1} + \hat{\boldsymbol{r}}\hat{\boldsymbol{r}})r^{-1}, \tag{5.7}$$

$$\int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) = 2\pi^2 \frac{\partial^2}{\partial \boldsymbol{r}^2} r^{-1}. \tag{5.8}$$

* This was asserted in paper I from an incomplete argument.

In the above equations $\hat{r}_{ij} \equiv R_{ij}/R_{ij}$ and $\hat{r} \equiv r/r$ are unit vectors; the arrow $\leftarrow$ on $\overleftarrow{\partial}/\partial R_{ij}$ in eq. (5.5) indicates a differentiation to the left.

In order to simplify the expression for H we note that since $(\partial/\partial r)\cdot(\partial/\partial r)r^{-1} = 0$, one obviously has

$$\overline{\frac{\partial^p}{\partial r^p}} r^{-1} = \frac{\partial^p}{\partial r^p} r^{-1} . \tag{5.9}$$

Using this identity together with the formula[21])

$$\frac{\partial^p}{\partial r^p} r^{-1} = (-1)^p (2p-1)!! r^{-(p+1)} \overline{\hat{r}^p} , \tag{5.10}$$

one obtains from eq. (5.6) the final result

$$H_{ij}^{(n,m)} = (-1)^m \frac{3}{4} a_i^n a_j^{m-1} \left(\frac{a_i^2}{2n+1} + \frac{a_j^2}{2m+1} \right) (2n+2m-1)!! R_{ij}^{-(n+m+1)} \overline{\hat{r}_{ij}^{n+m}} , \tag{5.11}$$

derived before in paper I. The differentiations in expression (5.5) for the tensor G may be carried out in a similar general way; for many purposes, however, the form (5.5) is as convenient.

The above results for the connectors $A_{ij}^{(n,m)}$ are valid only for $i \neq j$; for the case $i = j$, general explicit expressions for the tensor $B^{(n,n)} \equiv -A_{ii}^{(n,n)}$ (and for its inverse) have been obtained in ref. 7.

The evaluation of the wall connectors W (defined in eq. (3.9)) proceeds along the same lines as the evaluation of A given above. Here too, only a few terms in the expansion of the Bessel functions in (3.9) in powers of k give a nonzero contribution upon integration. After some algebra (cf. appendix A) one obtains the result (valid *also* for $i = j$)

$$W_{ij}^{(n,m)} = {}^{a}W_{ij}^{(n,m)} + {}^{b}W_{ij}^{(n,m)} + {}^{c}W_{ij}^{(n,m)} , \tag{5.12}$$

with the definitions

$${}^{a}W_{ij}^{(n\,m)} = (-1)^n \frac{3}{4} a_i^n a_j^{m-1} \left[\overline{\frac{\partial^{n-1}}{\partial R_{ij_s}^{n-1}}} \left(S - 2l_i \frac{\partial}{\partial R_{ij_s}} \hat{n} \right) + 2(n-1)\, \hat{n} \overline{\frac{\partial^{n-2}}{\partial R_{ij_s}^{n-2}}} \frac{\partial}{\partial R_{ij_s}} \hat{n} \right] \cdot (\mathbf{1} + \hat{r}_{ij_s}\hat{r}_{ij_s}) R_{ij_s}^{-1} \cdot S \overline{\frac{\overleftarrow{\partial}^{m-1}}{\partial (S \cdot R_{ij_s})^{m-1}}}$$

$$+(-1)^m \frac{3}{2} a_i^n a_j^{m-1} R_{ij_s}^{-(n+m-1)} \Big[(2n+2m-1)!!(l_i/R_{ij_s})^2 \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m}}$$
$$+ 2(n-1)(2n+2m-3)!! l_i R_{ij_s}^{-1} \boldsymbol{\Delta}^{(n-1\, n-1)} \odot \hat{\boldsymbol{n}} \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m-1}}$$
$$+ (n-1)(n-2)(2n+2m-5)!! \boldsymbol{\Delta}^{(n-1\, n-1)} \odot \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m-2}} \Big] \odot \boldsymbol{\Sigma}^{(m,m)}, \tag{5.13}$$

$${}^{b}\boldsymbol{W}_{ij}^{(n\, m)} = (-1)^n \frac{3}{2} a_i^{n+2} a_j^{m-1} (2n+1)^{-1} \frac{\overline{\partial^{n-1}}}{\partial \boldsymbol{R}_{ij_s}^{n-1}} \frac{\partial^2}{\partial \boldsymbol{R}_{ij_s}^2} \cdot \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} \cdot (\boldsymbol{1} + \hat{\boldsymbol{r}}_{ij_s} \hat{\boldsymbol{r}}_{ij_s}) R_{ij_s}^{-1}$$
$$\cdot \boldsymbol{S} \overline{\frac{\overleftarrow{\partial}^{m-1}}{\partial(\boldsymbol{S} \cdot \boldsymbol{R}_{ij_s})^{m-1}}} + (-1)^m \frac{3}{2} a_i^n a_j^{m-1} R_{ij_s}^{-(n+m+1)} (2n+2m-1)!!$$
$$\times \Big[(2n+1)^{-1} a_i^2 \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m}} \odot \boldsymbol{\Sigma}^{(m,m)} - \frac{1}{2} \Big(\frac{a_i^2}{2n+1} + \frac{a_j^2}{2m+1} \Big) \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m}} \odot \boldsymbol{\Sigma}^{(m+1,m+1)}$$
$$+ \Big(\frac{a_i^2}{2n+1} - \frac{a_j^2}{2m+1} \Big) ((2n+2m+1) l_i R_{ij_s}^{-1} \hat{\boldsymbol{n}} \cdot \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m+1}} \odot \boldsymbol{\Sigma}^{(m,m)}$$
$$+ (n-1) \boldsymbol{\Delta}^{(n-1,n-1)} \odot \hat{\boldsymbol{n}} (\hat{\boldsymbol{n}} \cdot \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m}}) \odot \boldsymbol{\Sigma}^{(m,m)}) \Big], \tag{5.14}$$

$${}^{c}\boldsymbol{W}_{ij}^{(n,m)} = (-1)^{m+1} \frac{3}{2} a_i^{n+2} a_j^{m+1} R_{ij_s}^{-(n+m+3)} (2n+2m+3)!!(2n+1)^{-1}(2m+1)^{-1}$$
$$\times \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} : \overline{\hat{\boldsymbol{r}}_{ij_s}^{n+m+2}} \odot \boldsymbol{\Sigma}^{(m,m)}. \tag{5.15}$$

In the above equations the vector $\boldsymbol{R}_{ij_s} \equiv \boldsymbol{S} \cdot \boldsymbol{R}_j - \boldsymbol{R}_i$ points from sphere i to the mirror image with respect to the wall of sphere j (the reflection tensor $\boldsymbol{S}$ was defined in eq. (2.9)); this vector has magnitude $R_{ij_s} = (R_{ij}^2 + 4 l_i l_j)^{1/2}$ (where $l_i \equiv \hat{\boldsymbol{n}} \cdot \boldsymbol{R}_i$ and $l_j \equiv \hat{\boldsymbol{n}} \cdot \boldsymbol{R}_j$ denote the distances of spheres i and j to the wall) and direction $\hat{\boldsymbol{r}}_{ij_s} \equiv \boldsymbol{R}_{ij_s}/R_{ij_s}$. The tensor $\boldsymbol{\Sigma}^{(n,n)}$ of rank $2n$ is an n-fold ordered product of the tensor $\boldsymbol{S}$,

$$\Sigma^{(n,n)}_{\mu_1 \ldots \mu_n, \nu_1 \ldots \nu_n} \equiv S_{\mu_1 \nu_n} S_{\mu_2 \nu_{n-1}} \ldots S_{\mu_n \nu_1}. \tag{5.16}$$

The tensor $\boldsymbol{\Delta}^{(n,n)}$ of rank $2n$ is a tensor which projects out the irreducible part of a tensor of rank n:

$$\boldsymbol{\Delta}^{(n,n)} \odot \boldsymbol{b}^n = \boldsymbol{b}^n \odot \boldsymbol{\Delta}^{(n,n)} = \overline{\boldsymbol{b}^n}. \tag{5.17}$$

For $n = 1, 2$ one has e.g.[21])

$$\Delta^{(1,1)}_{\mu\nu} = \delta_{\mu\nu}, \quad \Delta^{(2,2)}_{\mu\nu\kappa\lambda} = \tfrac{1}{2}\delta_{\mu\kappa}\delta_{\nu\lambda} + \tfrac{1}{2}\delta_{\nu\lambda}\delta_{\mu\kappa} - \tfrac{1}{3}\delta_{\mu\nu}\delta_{\kappa\lambda}. \tag{5.18}$$

The dot $\odot$ is used in eqs. (5.13)–(5.15) in conjunction with the tensors $\boldsymbol{\Sigma}^{(n,n)}$ and $\boldsymbol{\Delta}^{(n,n)}$ ($n = 1, 2, \ldots$) defined above, to denote an n-fold contraction (with the nesting convention, cf. eq. (3.10)).

Substitution of the above expressions for the connectors in equations (4.8)–(4.11) yields the mobilities, or (with eq. (4.17)) the fluid velocity field as an expansion in the two parameters a/R and $a/R_s \equiv a/(R^2 + 4l^2)^{1/2}$; here a is the typical radius of a sphere and R and l are the typical distances between two spheres and between a sphere and the wall, respectively. The dependence of the connectors on these two parameters is as follows*:

$$\begin{aligned} &\boldsymbol{G}^{(n,m)} \propto (a/R)^{n+m-1}, \quad \boldsymbol{H}^{(n,m)} \propto (a/R)^{n+m+1}, \quad {}^{\mathrm{a}}\boldsymbol{W}^{(n,m)} \propto (a/R_s)^{n+m-1}, \\ &{}^{\mathrm{b}}\boldsymbol{W}^{(n,m)} \propto (a/R_s)^{n+m+1}, \quad {}^{\mathrm{c}}\boldsymbol{W}^{(n,m)} \propto (a/R_s)^{n+m+3}, \end{aligned} \tag{5.19}$$

hence products of connectors with small upper indices n and m give the dominant contributions.

6. Explicit results

Equations (4.8)–(4.11) and (4.17) together with the expressions for the connectors $\boldsymbol{A}_{ij}^{(n,m)}$ ($i \neq j$), $\boldsymbol{W}_{ij}^{(n,m)}$ and $\boldsymbol{B}^{(n,n)^{-1}}$ given in section 5 and in ref. 7 enable one to calculate the (translational and rotational) mobility tensors, as well as the fluid velocity field, to any desired order in the two parameters a/R and $a/R_s \equiv a/(R^2 + 4l^2)^{1/2}$.

In paper I the mobilities in an unbounded fluid were evaluated explicitly up to and including terms of order $(a/R)^7$. To this order hydrodynamic interactions between two, three and four spheres contribute. The fluid velocity field in an unbounded fluid to this order follows directly from these results, by virtue of eq. (4.17). We shall give below explicit expressions for the mobilities in the presence of a wall, including terms of order $(a/R)^n (a/R_s)^m$ with $n + m \leqslant 3$. To this order specific hydrodynamic interactions of one and two spheres and the wall contribute. One finds:

$$\begin{aligned} 6\pi\eta a_i \boldsymbol{\mu}_{ij}^{\mathrm{TT}} &= \boldsymbol{1}\delta_{ij} + [\boldsymbol{G}_{ij}^{(1,1)} + \boldsymbol{H}_{ij}^{(1,1)}](1 - \delta_{ij}) + {}^{\mathrm{a}}\boldsymbol{W}_{ij}^{(1,1)} + {}^{\mathrm{b}}\boldsymbol{W}_{ij}^{(1,1)} \\ &= \boldsymbol{1}\delta_{ij} + [\tfrac{3}{4} a_i R_{ij}^{-1}(\boldsymbol{1} + \hat{\boldsymbol{r}}_{ij}\hat{\boldsymbol{r}}_{ij}) - \tfrac{3}{4} a_i (a_i^2 + a_j^2) R_{ij}^{-3}(\hat{\boldsymbol{r}}_{ij}\hat{\boldsymbol{r}}_{ij} - \tfrac{1}{3}\boldsymbol{1})](1 - \delta_{ij}) \\ &\quad - \tfrac{3}{4} a_i R_{ij_s}^{-1}[\boldsymbol{1} + \hat{\boldsymbol{r}}_{ij_s}\hat{\boldsymbol{r}}_{i_s j} - 2 l_i R_{ij_s}^{-1}\hat{\boldsymbol{r}}_{ij_s}\hat{\boldsymbol{n}} + 2 l_j R_{ij_s}^{-1}\hat{\boldsymbol{n}}\hat{\boldsymbol{r}}_{i_s j} + 2 l_i l_j R_{ij_s}^{-2} \end{aligned}$$

* The wall connector $\boldsymbol{W}_{ij}$ may also refer to a single sphere, which is the case if $i = j$ The variable R_{ij_s} then reduces to $2l_i$ and the expansion parameter a/R_s is therefore $\frac{1}{2}a/l$ in this case

$$\times(\mathbf{1}-2\hat{n}\hat{n}-3\hat{r}_{ij_s}\hat{r}_{i_sj})]+\tfrac{3}{4}a_i(a_i^2+a_j^2)R_{ij_s}^{-3}(\hat{r}_{ij_s}\hat{r}_{i_sj}-\tfrac{1}{3}\mathbf{1})$$
$$-\tfrac{3}{2}a_i(a_i^2-a_j^2)R_{ij_s}^{-4}(l_i\hat{r}_{ij_s}\hat{n}+l_j\hat{n}\hat{r}_{i_sj})+\tfrac{3}{2}a_iR_{ij_s}^{-5}(a_i^2l_j+a_j^2l_i)$$
$$\times(l_i+l_j)(\mathbf{1}-2\hat{n}\hat{n}-5\hat{r}_{ij_s}\hat{r}_{i_sj}), \tag{6.1}$$

$$8\pi\eta a_i^2a_j\boldsymbol{\mu}_{ij}^{\mathrm{RR}}=\mathbf{1}\delta_{ij}-\tfrac{1}{3}\boldsymbol{\epsilon}:[\boldsymbol{G}_{ij}^{(2a,2a)}(1-\delta_{ij})+{}^{a}\boldsymbol{W}_{ij}^{(2a,2a)}]:\boldsymbol{\epsilon}$$
$$=\mathbf{1}\delta_{ij}+\tfrac{3}{2}a_i^2a_jR_{ij}^{-3}(\hat{r}_{ij}\hat{r}_{ij}-\tfrac{1}{3}\mathbf{1})(1-\delta_{ij})-\tfrac{3}{2}a_i^2a_jR_{ij_s}^{-3}[\hat{r}_{i_sj}\hat{r}_{ij_s}-\tfrac{1}{3}\mathbf{1}$$
$$+2(l_i+l_j)^2R_{ij_s}^{-2}\mathbf{1}-2(\hat{r}_{ij_s}\wedge\hat{n})(\hat{r}_{ij_s}\wedge\hat{n})], \tag{6.2}$$

$$12\pi\eta a_i^2\boldsymbol{\mu}_{ij}^{\mathrm{RT}}=12\pi\eta a_i^2\tilde{\boldsymbol{\mu}}_{ji}^{\mathrm{TR}}=-\boldsymbol{\epsilon}:[\boldsymbol{G}_{ij}^{(2a,1)}(1-\delta_{ij})+{}^{a}\boldsymbol{W}_{ij}^{(2a,1)}]$$
$$=-\tfrac{3}{2}a_i^2R_{ij}^{-2}\boldsymbol{\epsilon}\cdot\hat{r}_{ij}(1-\delta_{ij})+\tfrac{3}{2}a_i^2R_{ij_s}^{-2}[\boldsymbol{\epsilon}\cdot\hat{r}_{ij_s}+2l_jR_{ij_s}^{-1}(\boldsymbol{\epsilon}\cdot\hat{n}-3(\hat{r}_{ij_s}\wedge\hat{n})\hat{r}_{i_sj})]. \tag{6.3}$$

We list below the notations used:

$$R_{ij}\equiv|\boldsymbol{R}_j-\boldsymbol{R}_i|,\quad l_i\equiv\hat{n}\cdot\boldsymbol{R}_i,\quad l_j\equiv\hat{n}\cdot\boldsymbol{R}_j,\quad \hat{r}_{ij}\equiv(\boldsymbol{R}_j-\boldsymbol{R}_i)/R_{ij},$$
$$R_{ij_s}\equiv|\boldsymbol{S}\cdot\boldsymbol{R}_j-\boldsymbol{R}_i|=(R_{ij}^2+4l_il_j)^{1/2},\quad \hat{r}_{ij_s}\equiv(\boldsymbol{S}\cdot\boldsymbol{R}_j-\boldsymbol{R}_i)/R_{ij_s},$$
$$\hat{r}_{i_sj}\equiv(\boldsymbol{R}_j-\boldsymbol{S}\cdot\boldsymbol{R}_i)/R_{ij_s}=\hat{r}_{ij_s}+2\hat{n}(l_i+l_j)/R_{ij_s};$$

the unit tensor and the Levi–Civita tensor are denoted by $\mathbf{1}$ and $\boldsymbol{\epsilon}$ respectively; $\tilde{\boldsymbol{\mu}}$ denotes the transposed of $\boldsymbol{\mu}$.

The expressions for the mobilities given above reduce for $i=j$ to the well-known results[1,14,15]) to order $(a/l)^3$ for a single sphere at a distance l from a plane wall*. Note that to this order there is no coupling between translation and rotation for a single sphere ($\boldsymbol{\mu}_{ii}^{\mathrm{RT}}$ is in fact of order $(a/l)^4$).

The fluid velocity field follows directly from eqs. (6.1) and (6.3) by virtue of relation (4.17). One then finds to third order the fluid velocity field due to the motion of a single sphere in the presence of a wall.

The calculation of higher order contributions to the mobilities and the fluid velocity field is within the present scheme elementary, in the sense that only differentiations and tensor contractions are required. To fifth order e.g., $\boldsymbol{\mu}_{ij}^{\mathrm{TT}}$ is given by

$$6\pi\eta a_i\boldsymbol{\mu}_{ij}^{\mathrm{TT}}=\mathbf{1}\delta_{ij}+\boldsymbol{A}_{ij}^{(1,1)}(1-\delta_{ij})+\boldsymbol{W}_{ij}^{(1,1)}$$
$$+\sum_{k=1}^{N}(\boldsymbol{G}_{ik}^{(1,2s)}(1-\delta_{ik})+{}^{a}\boldsymbol{W}_{ik}^{(1,2s)}):\boldsymbol{B}^{(2s,2s)^{-1}}:(\boldsymbol{G}_{kj}^{(2s,1)}(1-\delta_{kj})+{}^{a}\boldsymbol{W}_{kj}^{(2s,1)}) \tag{6.4}$$

* We have in fact verified to order $(a/l)^5$ the agreement of our expressions for a single sphere with results from the litterature

and contains specific hydrodynamic interactions of up to three spheres and the wall. The explicit expression corresponding to eq. (6.4) is however very lengthy and will not be recorded here.

7. Concluding remarks

Using a result due to Lorentz[18,19]) we have extended the scheme developed in paper I[10]) – to evaluate the mobility tensors for an arbitrary number of spheres in an unbounded fluid – to include the presence of a plane wall. The fluid velocity field can be obtained from these mobilities by a simple relation (eq. (4.17)). Friction tensors, on the other hand, may be found by inversion of the mobility tensor matrix – or more directly from the hierarchy of equations (3.7). In appendix B we give the expressions for the translational and rotational friction tensors, and consider also the case of freely rotating spheres.

The friction tensors for a system of two spheres and a plane wall have been studied by Wakiya[16]), for the case of two non-rotating spheres moving with equal velocities in the plane which is perpendicular to the wall and passes through the centers of both spheres (that is to say $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = 0$; $\boldsymbol{u}_1 = \boldsymbol{u}_2 \equiv \boldsymbol{u}$; $\boldsymbol{u}$, $\hat{\boldsymbol{n}}$ and $\boldsymbol{R}_{12}$ coplanar). His explicit expressions (to lowest order) agree with those resulting from the general formulae in appendix B. Vasseur and Cox[17]) have investigated the lift forces (i.e. the components of the forces perpendicular to the wall) on two freely rotating spherical particles, moving in the same direction parallel to a plane wall. They included in their analysis the effect of the non-linear convection term in the Navier–Stokes equation. We have verified that the results of Vasseur and Cox agree with ours in the limit of zero Reynolds number.

In our treatment we have assumed that the unperturbed fluid is at rest (cf. eq. (2.26)). The generalization to the case that $\boldsymbol{v}_0$ is an arbitrary non-vanishing solution of the quasi-static Stokes equation (with $\boldsymbol{v}_0 = 0$ on the wall) is, however, straightforward – as pointed out in paper I (section 7) for the case of an unbounded fluid*.

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* In the third term on r.h.s. of eq. (I-7.4) a factor $a_j a_i^{-1}$ has been left out.

Appendix A. Evaluation of the wall connectors

Upon partial integration eq. (3.9) for the wall connector $\boldsymbol{W}$ may be written as

$$\boldsymbol{W}_{ij}^{(n,m)} = -\tfrac{3}{4}\pi^{-2} a_i (-i)^{m-n}(2n-1)!!(2m-1)!! \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{-i\boldsymbol{k}\cdot(\boldsymbol{S}\cdot\boldsymbol{R}_j - \boldsymbol{R}_i)} j_{m-1}(a_j k)$$

$$\times \Big\{ k^{-2} j_{n-1}(a_i k)\, \overline{\hat{\boldsymbol{k}}^{n-1}}\, (\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\, \overline{\hat{\boldsymbol{k}}^{m-1}} \odot \boldsymbol{\Sigma}^{(m+1,m+1)}$$

$$+ 2k^{-1}\hat{\boldsymbol{n}} \cdot \Big[\Big(\frac{\partial}{\partial \boldsymbol{k}} + \mathrm{i}\boldsymbol{R}_i\Big) j_{n-1}(a_i k)\, \overline{\hat{\boldsymbol{k}}^{n-1}}\Big] \hat{\boldsymbol{k}}\hat{\boldsymbol{n}} \cdot (\boldsymbol{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\, \overline{\hat{\boldsymbol{k}}^{m-1}} \odot \boldsymbol{\Sigma}^{(m,m)}$$

$$+ \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} : \Big[\Big(\frac{\partial}{\partial \boldsymbol{k}}\frac{\partial}{\partial \boldsymbol{k}} + 2\mathrm{i}\boldsymbol{R}_i \frac{\partial}{\partial \boldsymbol{k}} - \boldsymbol{R}_i\boldsymbol{R}_i\Big) j_{n-1}(a_i k)\, \overline{\hat{\boldsymbol{k}}^{n-1}}\Big] \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \overline{\hat{\boldsymbol{k}}^{m-1}} \odot \boldsymbol{\Sigma}^{(m,m)} \Big\}, \tag{A.1}$$

where we have used the formula

$$\overline{(\boldsymbol{S}\cdot\hat{\boldsymbol{k}})^m} = \overline{\hat{\boldsymbol{k}}^m} \odot \boldsymbol{\Sigma}^{(m,m)} \tag{A.2}$$

(with the tensor $\boldsymbol{\Sigma}$ defined in eq. (5.16)). The dot in e.g. $\odot\,\boldsymbol{\Sigma}^{(m,m)}$ denotes an m-fold contraction (with the nesting convention, cf. eq. (3.10)). To perform the differentiations between square brackets in eq. (A.1) the following formulae are helpful

$$\hat{\boldsymbol{n}} \cdot \frac{\partial}{\partial \boldsymbol{k}} \overline{\hat{\boldsymbol{k}}^{p-1}} = k^{-1}(p-1)\big[\, \overline{\hat{\boldsymbol{n}}\hat{\boldsymbol{k}}^{p-2}} - (\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}})\, \overline{\hat{\boldsymbol{k}}^{p-1}}\,\big], \tag{A.3}$$

$$\hat{\boldsymbol{n}} \cdot \frac{\partial}{\partial \boldsymbol{k}}\, \hat{\boldsymbol{n}} \cdot \frac{\partial}{\partial \boldsymbol{k}} \overline{\hat{\boldsymbol{k}}^{p-1}} = k^{-2}(p-1)\big[((p+1)(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}})^2 - 1)\, \overline{\hat{\boldsymbol{k}}^{p-1}} - 2(p-1)(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}})\, \overline{\hat{\boldsymbol{n}}\hat{\boldsymbol{k}}^{p-2}}$$

$$+ (p-2)\, \overline{\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\hat{\boldsymbol{k}}^{p-3}}\,\big]. \tag{A.4}$$

By virtue of the Paley–Wiener theorem[25]), only a few terms in the expansion of the Bessel functions in eq. (A.1) in powers of k give a nonzero contribution upon integration; these remaining contributions may be evaluated with the help of formulae (5.7)–(5.10). Collecting terms of the same order in $R_{ij_s} \equiv |\boldsymbol{S}\cdot\boldsymbol{R}_j - \boldsymbol{R}_i|$ gives the result (5.12)–(5.15).

Appendix B. Friction tensors

The friction tensors are defined as follows ($i = 1, 2, \dots, N$)

$$\begin{aligned} \boldsymbol{K}_i &= -\sum_{j=1}^{N} (\zeta_{ij}^{\mathrm{TT}} \cdot \boldsymbol{u}_j + \zeta_{ij}^{\mathrm{TR}} \cdot \boldsymbol{\omega}_j) , \\ \boldsymbol{T}_i &= -\sum_{j=1}^{N} (\zeta_{ij}^{\mathrm{RT}} \cdot \boldsymbol{u}_j + \zeta_{ij}^{\mathrm{RR}} \cdot \boldsymbol{\omega}_j) . \end{aligned} \tag{B.1}$$

By elimination of higher order multipoles of the induced force in the hierarchy of eqs. (3.7) one obtains for these tensors the expressions

$$(6\pi\eta a_j)^{-1}\zeta_{ij}^{\mathrm{TT}} = \boldsymbol{1}\delta_{ij} - \boldsymbol{C}_{ij}^{(1,1)} - \sum_{s=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_s=1}^{\infty}\sum_{j_1=1}^{N}\cdots\sum_{j_s=1}^{N} \boldsymbol{C}_{ij_1}^{(1,m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \dots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,1)} , \tag{B.2}$$

$$(8\pi\eta a_i a_j^2)^{-1}\zeta_{ij}^{\mathrm{RR}} = \boldsymbol{1}\delta_{ij} + \tfrac{1}{3}\boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2\mathrm{a},2\mathrm{a})} : \boldsymbol{\epsilon} + \tfrac{1}{3}\sum_{s=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_s=1}^{\infty}\sum_{j_1=1}^{N}\cdots\sum_{j_s=1}^{N} \boldsymbol{\epsilon} : \boldsymbol{C}_{ij_1}^{(2\mathrm{a},m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \dots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,2\mathrm{a})} : \boldsymbol{\epsilon} , \tag{B.3}$$

$$(4\pi\eta a_i a_j)^{-1}\zeta_{ij}^{\mathrm{RT}} = \boldsymbol{\epsilon} : \boldsymbol{C}_{ij}^{(2\mathrm{a},1)} + \sum_{s=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_s=1}^{\infty}\sum_{j_1=1}^{N}\cdots\sum_{j_s=1}^{N} \boldsymbol{\epsilon} : \boldsymbol{C}_{ij_1}^{(2\mathrm{a},m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \dots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,1)} , \tag{B.4}$$

$$(4\pi\eta a_j^2)^{-1}\zeta_{ij}^{\mathrm{TR}} = -\boldsymbol{C}_{ij}^{(1,2\mathrm{a})} : \boldsymbol{\epsilon} - \sum_{s=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_s=1}^{\infty}\sum_{j_1=1}^{N}\cdots\sum_{j_s=1}^{N} \boldsymbol{C}_{ij_1}^{(1,m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{C}_{j_1j_2}^{(m_1,m_2)} \odot \dots \odot \boldsymbol{B}^{(m_s,m_s)^{-1}} \odot \boldsymbol{C}_{j_sj}^{(m_s,2\mathrm{a})} : \boldsymbol{\epsilon} , \tag{B.5}$$

with the convention

$$\boldsymbol{C}_{ij}^{(n,2)} : \boldsymbol{B}^{(2,2)^{-1}} \equiv -\tfrac{10}{9}\boldsymbol{C}_{ij}^{(n,2\mathrm{s})} - \tfrac{2}{3}\boldsymbol{C}_{ij}^{(n,2\mathrm{a})} \tag{B.6}$$

(cf. eqs. (4.5) and (4.12)). The above equations are the analogues for friction tensors of eqs. (4.8)–(4.11) for mobilities.

It is evident from the above expressions (using property (5.19) of the connectors) that the dominant n-sphere contributions to the friction tensors are of order $(a/R)^{n-1}$ and are due to sequences $\boldsymbol{C}^{(1,1)} \cdot \boldsymbol{C}^{(1,1)} \cdot \dots \cdot \boldsymbol{C}^{(1,1)}$, i.e. to monopole–monopole interactions. In contrast, the dominant n-sphere contributions to the mobilities are (as noted in paper I) of the higher order $(a/R)^{3n-5}$ and are due to sequences $\boldsymbol{C}^{(1\,2s)}\cdot\boldsymbol{C}^{(2s\,2s)}\cdot \quad \boldsymbol{C}^{(2s\,1)}$, i e (essentially) to dipole–dipole interactions

To conclude our discussion of friction tensors we consider freely rotating spheres In this case the torques on the spheres are zero and one can eliminate their angular velocities to give

$$\boldsymbol{K}_i = -\sum_{j=1}^{N} \zeta_{ij}^{\mathrm{F}}\cdot\boldsymbol{u}_j \tag{B 7}$$

The free-rotation friction tensor ζ_{ij}^{F} may be determined by inversion of $\boldsymbol{\mu}_{ij}^{\mathrm{TT}}$ (this was done to order $(a/R)^3$ in ref 9), or directly from the hierarchy (3 7) The resulting expression for ζ_{ij}^{F} in terms of the connectors is identical to expression (B 2) for ζ_{ij}^{TT}, with the proviso that convention (B 6) is now replaced by

$$\boldsymbol{C}_{ij}^{(n\,2)}\ \boldsymbol{B}^{(2\,2)\,-1} \equiv -\tfrac{10}{9}\boldsymbol{C}_{ij}^{(n\,2s)}, \tag{B 8}$$

excluding the antisymmetric part

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