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COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On the secular accelerations and the fluctuations of the longitudes of the moon, the sun, Mercury and Venus, by *W. de Sitter*.

1. *Introduction.*

Several investigators have lately found in the longitudes of the sun and the planets fluctuations which run parallel to those discovered by NEWCOMB in the moon's longitude. Their results, however, while agreeing in establishing the reality of these fluctuations, disagree on two important points. One of these is whether the fluctuations of the longitude of the sun and the planets are equal to those of the moon diminished in the exact ratio of the mean motions, or to those fluctuations so diminished multiplied by a factor, for which values ranging from unity to about 2·5 are found. The second questionable point is whether the fluctuations of the sun and planets agree with the total fluctuations of the moon, or with the 'minor fluctuations' which remain after the removal of the 'great empirical term'.

It is very desirable to decide these two points by a discussion of the observations alone, unbiassed by any reference to a possible explanation which might favour one or another particular answer to the questions raised.

For such a discussion it is necessary in the longitude of each heavenly body to introduce as unknowns corrections to the mean motion and the longitude at the epoch, and also a secular acceleration of the mean longitude. The corrections to the other elements can be neglected at the present stage. In the case of Mercury, however, it is necessary to take account of the equation of the centre making the correction to the true longitude different from that to the mean longitude.

To introduce the secular acceleration, we put

$$S = T^2 + 1{\cdot}33\ T - 0{\cdot}26\,,$$

the time T being counted in centuries from 1900·0. S is zero for 1750·0 and 1917·1. Further let B' be the total fluctuation of the moon's longitude. The excess of the moon's true longitude over that given by BROWN's tables is then

$$\Delta L = \Delta L_0 + T\Delta n + 5''{\cdot}92\,(1+\varkappa')\,S + B' - 10''{\cdot}71 \sin\,(140^\circ{\cdot}0\ T + 240^\circ{\cdot}7)$$

The correction to the longitude of the sun from NEWCOMB's tables is

$$\Delta L' = \Delta L_0' + T\Delta n_0 + S(1+\varkappa) + Q\frac{n_0}{n}B'\,,$$

and to the mean longitudes of Mercury and Venus

$$\Delta\lambda_i = \Delta\lambda_{i0} + T\Delta n_i + \frac{n_i}{n_0}S(1+\varkappa_i) + Q_i\frac{n_i}{n}B'.$$

As a first approximation I have taken $Q = Q_i = 1{\cdot}50$.

The values of $1{\cdot}50\ n_i/n$ and of n_i/n_0 are

Sun	$1{\cdot}50\ n_0/n = 0{\cdot}112$	
Mercury ($i=1$)	$1{\cdot}50\ n_1/n = 0{\cdot}466$	$n_1/n_0 = 4{\cdot}15$
Venus ($i=2$)	$1{\cdot}50\ n_2/n = 0{\cdot}183$	$n_2/n_0 = 1{\cdot}63$

There are thus in all four unknowns for the sun and for each of the planets, and three for the moon. It is not to be expected, however, that it will be found possible to determine all fifteen unknowns independently of each other. Especially the $\varkappa_i$ can only be determined from a very long series of exact observations. It has been found that the transits of Mercury are the only series of observations allowing an independent determination of $\varkappa_1$.

2. *Secular accelerations of the sun and the moon.*

These can only be determined from the ancient observations of eclipses and other phenomena. For the moon the secular acceleration cannot be separated from the fluctuations in the modern observations, which cover only two centuries and a half, from 1670 to 1925. For the sun the only entirely trustworthy observations are those made after 1830 or 1835.

If we leave the fluctuations out of account, the excess of the true over the tabular longitude at the time T is

$$\textit{for the sun:}\quad \Delta L' = \Delta L_0' + T\Delta n_0 + S\,(1+\varkappa)$$
$$\textit{for the moon:}\quad \Delta L = \Delta L_0 + T\Delta n + 5''{\cdot}92\,S\,(1+\varkappa')$$

The ancient observations have been very thoroughly discussed by FOTHERINGHAM and SCHOCH. From these discussions I derive the following equations of condition for $\varkappa$ and $\varkappa'$.

a. From HIPPARCHUS's determinations of the equinox FOTHERINGHAM finds (*M.N.* **78**, p. 416) a correction to the sun's longitude of

$$\begin{aligned}\Delta L' &= +\,(1''{\cdot}95 \pm 0''{\cdot}27)\;T^2 + 3''{\cdot}9\;T + 1''{\cdot}95\\ &= +\,730'' \pm 125'',\end{aligned}$$

which is valid for the epoch $T = -20{\cdot}4$.

b. From solar eclipses FOTHERINGHAM finds (*M.N.* **81**, p. 126) for the mean epoch $T = -21$

$$\Delta L' = +\,(1''{\cdot}5 \pm 0''{\cdot}3)\;T^2 + 3''{\cdot}0\,T + 1''{\cdot}5 = +600'' \pm 130''.$$

The probable error has been estimated from the diagram given *l.c.* p. 123.

c. From the magnitudes of lunar eclipses FOTHERINGHAM finds (*M.N.* **61**; p. 124, **69**, p. 668; **78**, p. 422) for the epoch $T = -18{\cdot}8$

$$\Delta L' = +\,570'' \pm 140''$$

d. From occultations FOTHERINGHAM finds (*M.N.* **75**, pp. 377 and 395) a correction to the moon's longitude of *)

$$\Delta L = +\,(4''{\cdot}8 \pm 0''{\cdot}7)\,T^2 + 10''{\cdot}7\;T + 1''{\cdot}2 = +2240'' \pm 270'',$$

which is valid for the epoch $T = -19{\cdot}6$.

e. From solar eclipses FOTHERINGHAM finds (*M.N.* **81**, p. 126) for $T = -21$

$$\Delta L = +\,1900'' \pm 220''$$

f. The most reliable solar eclipse, in fact the only one which was evidently observed according to a prearranged (and remarkably well arranged!) plan, is that known as the 'eclipse of HIPPARCHUS'. From FOTHERINGHAM's discussion of this eclipse in *M.N.* **69**, p. 208 and **81**, p. 122, we can derive an equation of condition between the two coordinates of the diagram (*M.N.* **81**, p. 123) which is

$$x - 2y = 8{\cdot}1 \pm 0{\cdot}2,$$

the probable error being estimated from the diagram.

The coordinates are determined by

$$\begin{aligned}\Delta L' &= (1) + (y - 1{\cdot}10)\;[(3) - (1)]\\ \Delta L &= (1) + (x - 10{\cdot}1\;)\;[(2) - (1)],\end{aligned}$$

where (1), (2), (3) are the three hypotheses on which FOTHERINGHAM carried out his computations, viz:

$$\begin{array}{lll} (1) & \Delta L' = +\,410'' & \Delta L = +\,1334''\\ (2) & \Delta L' = +\,410 & \Delta L = +\,1712\\ (3) & \Delta L' = +\,780 & \Delta L = +\,1334. \end{array}$$

The equation thus becomes

$$\Delta L' - 0{\cdot}49\;\Delta L = -\,280'' \pm 40''.$$

The epoch is $-20{\cdot}3$.

g. The results derived by FOTHERINGHAM (*M.N.* **80**, p. 580) from the observed times of lunar eclipses have been combined into two means, viz:

$$\begin{array}{ll} T = -20{\cdot}7 & \Delta L - \Delta L' = +\,2950'' \pm 660''\\ T = -17{\cdot}7 & \Delta L - \Delta L' = +\,2320 \;\pm 560''. \end{array}$$

h. Dr. SCHOCH *) has on the whole discussed the same eclipses as Dr. FOTHERINGHAM, and he derives very much the same secular accelerations. The most important of his eclipses not included in FOTHERINGHAM's discussion is a very precise observation of a lunar eclipse at Babylon in the year -424, of which the probable error may be estimated at $\pm\,7$ minutes of time. From this I derive, for $T = -23{\cdot}24$:

$$\Delta L - \Delta L' = +\,1700'' \pm 200''.$$

From these results I derive the following equations of condition:

$$\begin{array}{lllllllll}
(a) & \Delta L_0' & -20{\cdot}4\;\Delta n_0 & & & +\;389\;\varkappa & & = +\,340 & \pm\,125\\
(b) & \Delta L_0' & -21{\cdot}0\;\Delta n_0 & & & +\;413\;\varkappa & & = +\,187 & \pm\,130\\
(c) & \Delta L_0' & -18{\cdot}8\;\Delta n_0 & & & +\;328\;\varkappa & & = +\,240 & \pm\,140\\
(d) & \Delta L_0 & -19{\cdot}6\;\Delta n & & & & +2120\;\varkappa' & = +\,120 & \pm\,270\\
(e) & \Delta L_0 & -21{\cdot}0\;\Delta n & & & & +2445\;\varkappa' & = -\,545 & \pm\,220\\
(f) & \Delta L_0' - 0{\cdot}5\;\Delta L_0 & -20{\cdot}3\;\Delta n_0 & +10{\cdot}1\;\Delta n & & +\;384\;\varkappa & -1140\;\varkappa' & = +\,470 & \pm\;\;40\\
(g_1) & \Delta L_0' - \Delta L_0 & -20{\cdot}7\;\Delta n_0 & +20{\cdot}7\;\Delta n & & +\;400\;\varkappa & -2370\;\varkappa' & = -\,980 & \pm\,660\\
(g_2) & \Delta L_0' - \Delta L_0 & -17{\cdot}7\;\Delta n_0 & +17{\cdot}7\;\Delta n & & +\;290\;\varkappa & -1710\;\varkappa' & = -\,900 & \pm\,560\\
(h) & \Delta L_0' - \Delta L_0 & -23{\cdot}2\;\Delta n_0 & +23{\cdot}2\;\Delta n & & +\;520\;\varkappa & -3080\;\varkappa' & = +\,860 & \pm\,200
\end{array}$$

*) A correction of $-0''{\cdot}3$ to the coefficient of T^2 applied later (*M.N.* 83, p. 372) has been neglected, being far within the probable error.

*) *Die Seculäre Acceleration des Mondes und der Sonne*, Berlin, Dec. 1926. See also FOTHERINGHAM, *M.N.* 87, p. 154.

I now combine simular equations by weights, and reduce the resulting equations to a probable error of ± 0·10 of the right hand member. We then have:

								R_0	R_1	R_2
(a, b, c)	$+.00\ \Delta L_0'$	ΔL_0	$-\cdot 03\ \Delta n_0$	Δn	$+\cdot 50\ \varkappa$	$\varkappa'$	$= +\ \cdot 34$	$-\cdot 02$	$-\cdot 03$	$-\cdot 01$
(d, e)		$+\cdot 00$		$-\cdot 01$		$+1\cdot 01$	$= -\ \cdot 14$	$+\cdot 02$	$\cdot 00$	$+\cdot 03$
(f)	$+\cdot 00$	$-\cdot 00$	$-\cdot 05$	$+\cdot 03$	$+\cdot 96$	$-2\cdot 85$	$= +1\cdot 18$	$-\cdot 02$	$+\cdot 05$	$+\cdot 01$
(g)	$+\cdot 00$	$-\cdot 00$	$-\cdot 00$	$+\cdot 00$	$+\cdot 08$	$-\ \cdot 47$	$= -\ \cdot 22$	$-\cdot 36$	$-\cdot 33$	$[-\cdot 35]$
(h)	$+\cdot 00$	$-\cdot 00$	$-\cdot 01$	$+\cdot 01$	$+\cdot 26$	-1.54	$= +\ \cdot 43$	$-\cdot 01$	$+\cdot 02$	$\cdot 00$

It is clear that these equations can contribute nothing to the determination of $\Delta L_0'$, ΔL_0, Δn_0, and Δn, which must be deduced from modern observations. Taking as a first approximation $\Delta n_0 = +2$, $\Delta n = 0$, we find

$$\varkappa = +\cdot 85 \pm \cdot 16$$
$$\varkappa' = -\cdot 16 \pm \cdot 05,$$

the probable errors being derived from the weights, assuming a probable error of ± 0·10 for unit weight. These leave the residuals given under R_0.

From the discussion to be related below we find for the final values of Δn_0 and Δn

$$\Delta n_0 = +1\cdot 40 \qquad \Delta n = +4\cdot 00.$$

Introducing these we find from all equations:

$$\varkappa = +\cdot 826 \pm \cdot 160$$
$$\varkappa' = -\cdot 104 \pm \cdot 053,$$

leaving after substitution the residuals R_1.

If we reject the equation (g) we find

$$\varkappa = +\cdot 774 \pm \cdot 162$$
$$\varkappa' = -\cdot 128 \pm \cdot 053,$$

with the residuals R_2.

As the most probable values we may adopt *)

$$\varkappa = +0\cdot 80 \pm \cdot 16$$
$$\varkappa' = -0\cdot 12 \pm \cdot 05.$$

These correspond to the secular accelerations for

(1) *the sun*: $+ (1''\cdot 80 \pm 0''\cdot 16)\ S$
the moon: $+ (5\cdot 22 \pm \cdot 30)\ S$.

It is evident that, assuming the generally accepted ratio, the secular accelerations of the sun and the moon are contradictory, and apparently they cannot both be explained by a retardation of the earth's rotation by tidal friction. The probable errors of the equations of condition have been estimated rather too high than too low. Also the probable errors derived from the residuals would not be appreciably different, and they would be very much smaller if the equations (g) were rejected. From the transits of Mercury we found, as will be explained below, $\varkappa_1 = +0\cdot 55 \pm \cdot 10$, which is sufficiently near the value of $\varkappa$ for the sun not to exclude the possibility of the true values being identical. In that case we would be led to ascribe the whole of the secular acceleration of the sun and the planets to tidal friction, which would then by the usual ratio demand a secular acceleration of the moon nearly twice as large as found from the observations, i. e. of the same order as the *total* secular acceleration of the moon, *including* the part due to perturbations of the planets. It will be remenbered that LAPLACE originally explained the total observed value of $11''\ T^2$ in this way, leaving nothing for tidal friction: now we would be inclined to explain the greater part of it by tidal friction, leaving very little for perturbations.

A possible explanation of this apparent contradiction will be suggested below, in art. 7.

In the following discussions the values $\varkappa = +0\cdot 80$, $\varkappa' = -0\cdot 17$ were used, giving the secular accelerations $+1''\cdot 80\ S$ and $+4''\cdot 92\ S$ respectively. A correction corresponding to $\Delta L = +0''\cdot 30\ S$ has been applied to the final residuals.

*) See also footnote to page 36.

3. *The 'great empirical term' and the fluctuations.*

The question whether the fluctuations in the longitudes of other heavenly bodies correspond to the total, or to the minor, fluctuations in the moon, can only be solved by the discussion of a long series of observations. In the case of the sun and Venus we have practically only observations from 1835 to 1925 and during this interval the sinusoid representing the 'great empirical term' differs so little from a straight line, that any satisfactory representation of the total fluctuations can, by suitable corrections to the mean motion and the epoch, be transformed into an equally satisfactory representation of the minor fluctuations, or inversely, especially so since the value of Q must, if the distinction between the 'great empirical term' and the 'minor fluctuations' is to have any meaning at all, be assumed different for the two cases, which gives one more parameter by which to improve the representation of the observations.

The only series of observations which can be used for our purpose are the transits of Mercury across the sun's disc. We must thus derive an empirical sine term independently for the moon and for the transits of Mercury. If these two agree in period and phase, this proves that there is a correlation between the total fluctuations in the two cases. It does not follow that the sine term has a real existence, distinct from the total fluctuations. This would only be concluded if the remaining minor fluctuations either showed no correlation at all, or a correlation with a very different factor of proportionality.

The first step is thus to represent the total fluctuations B' of the moon's longitude by a formula

$$B_o' = \Delta L_o + T\Delta n + cS + K \sin(\beta T + \gamma).$$

For B' I have in this stage taken the values given by BROWN in his latest paper*) corrected for the difference of the adopted secular accelerations, thus

$$(2) \qquad B_o' = -(Th - G) - 0''.13\, S.$$

The residuals remaining after the substitution of the unknowns found from the solution are then the minor fluctuations F.

The term cS has been written down in the equation, although it was evident a priori, and also it appeared at once from the solution, that it is impossible to separate the determination of c from that of the elements of the sine term. Consequently c was taken equal to zero. For the sine term the values found for β and γ depend largely on the weights assigned to the early observations. The corrections found to the values adopted by BROWN, which were used as a first approximation, were not larger than the uncertainty of their determination. The distinction between the 'great empirical term' and the 'minor fluctuations' is rather arbitrary in any case, and for our purpose it is not very important which precise values are adopted for the period and the phase of the assumed sine term. I have therefore finally determined only the three unknowns ΔL_o, Δn and K from the equations of condition

$$B_o' = \Delta L_o + T\Delta n + K \sin(140^\circ.0\, T + 240^\circ.0).$$

Weights were assigned corresponding to an assumed probable error of unit weight of $\pm 0''.10$, the weight of a mean of three years modern Greenwich observations being 4 on this scale. The resulting values of the unknowns with their probable errors, computed from the weights, are:

$$\Delta L_o = +0''.66 \pm ''.02$$
$$\Delta n = +0.79 \pm .02$$
$$K = 14.42 \pm .03$$

The probable error of unit weight derived from the residuals F would be $\pm 3''.1$. The probable error of the unknowns corresponding to this value would be $\pm ''.50$, $\pm ''.70$ and $\pm ''.80$ respectively.

*) *Transactions of Yale University Observatory*, Vol. III, Part. 6.

Similarly for the transits of Mercury the equations of condition are:

$$O - C = a_1 + b_1 T + c_1 S + K_1 \sin(\beta_1 T + \gamma_1),$$

and the residuals remaining after the substitution are called F_1.

From INNES's discussion in *Union Observatory Circular* 65 we can derive the excess of the observed difference of true longitude of Mercury and the sun over the tabular value. In the case of Mercury a reduction from true to mean longitude is required on account of the large excentricity. It will be shown below that this can be effected with sufficient accuracy for our present purpose by using for the difference of true longitude the weights of the observed *times* and then treating the true longitudes as mean ones. It was again found impossible to determine the sine term independently of the secular acceleration. Consequently for c_1 the value corresponding to $\varkappa_1 = +0.80$ was taken. For the sine term we thus found

$$4''.57 \sin(136^\circ.0\, T + 236^\circ.4)$$
$$\pm .23 \qquad \pm 1.4 \qquad \pm 2.4$$

The agreement of the argument with that adopted for the moon is so close that there can be no reasonable doubt about their equality. I have therefore made a new solution introducing as unknowns only a_1, b_1 and K_1. The value found for the latter was

$$K_1 = 4''.39 \pm ''.21.$$

The ratio between this coefficient and the one found for the moon is

$$\frac{K_1}{K} = 0.304 = 1.32 \frac{n_1 - n_o}{n},$$

thus giving

$$(3) \qquad Q_1 = 1.32 \pm .07.$$

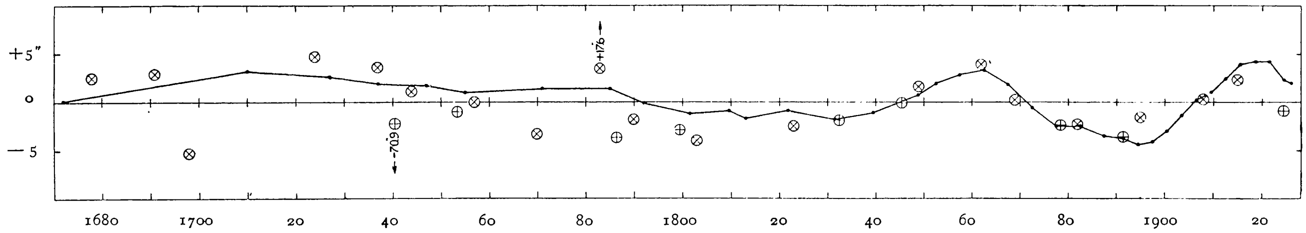
The residuals F_1, remaining after removing the sine term, must now be compared with the residuals F of the moon's longitude. This comparison is shown in figure 1, where, however, the final reduction of the transits of Mercury, which will be explained below, was used. The parallelism between them is at once apparent. A determination of the ratio gave

$$(3') \qquad \frac{F_1}{F} = (1.34 \pm .02) \frac{n_1 - n_o}{n}.$$

The probable error was derived from the residuals.

Although the uncertainty of the two values (3) and (3') of Q, is probably larger than is indicated by the probable errors, it is still very probable that the factors must be assumed to be the same for the sine term and for the residuals remaining after its removal.

FIGURE 1.



This proves that the distinction between the 'great empirical term' and the 'minor fluctuations' is entirely artificial, and it will henceforth be dropped. We will only consider the total fluctuations of the moon's longitude, which are now defined by

$$B' = B_o' - 0''\cdot66 - 0''\cdot79\ T,$$

B_o' being defined by (2). B' is thus the excess $O-C$ of the observed mean longitude O of the moon over the computed value C, which latter is

$$C = \text{BROWN's tables} - 10''\cdot71 \sin(140^\circ\cdot0\ T + 240^\circ\cdot7) + \\ + 4''\cdot79\ (T+1)^2 + 0''\cdot13\ S + 0''\cdot79\ T + 0''\cdot66.$$

The first three terms are the Th of BROWN's ($Th-G$), the last three are the correction found here. As has been mentioned above a further correction of $+ 0''\cdot30\ S$ was finally adopted. Including this we have

$$B' = \text{observed mean longitude of moon} - C$$

$$(4)\quad C = \text{BROWN's tables} - 10''\cdot71 \sin(140^\circ\cdot0\ T + 240^\circ\cdot7) \\ + 5''\cdot22\ S + 4''\cdot00\ T + 6''\cdot70.$$

For the observed longitude I have taken the results derived by NEWCOMB from eclipses and occultations up to 1835, and the Greenwich meridian observations after that date. The observed values were combined into normal places of five, four or three years, and were interpolated linearly between these for intermediate epochs. The values actually used are given in Table 1. These include the correction $+ 0''\cdot30\ S$ which was afterwards adopted, and do thus correspond exactly to the formula (4). The values for 1925·5 and 1926·5 were not used in the discussions, but were afterwards kindly communicated by the Astronomer Royal. The value for 1925·5 is, however, already contained in the normal place for 1924·5.

The probable error of the first normal point (a solar eclipse observed by GASSENDI) is ± 14″, of the second about ± 5″, and by 1681 it has come down to ± 1″. It then continues to decrease, being of the order of ± 0″·3 about 1800. The probable errors of the normal points after 1900 are ± 0″·04 or ± 0″·05.

The general shape of the curve shows some rough similarity to a sinusoid, as any variable quantity oscillating between limits and having the average value zero, must necessarily do. But the best fitting sinusoid leaves residuals over 4″. A much better fit can be obtained by a series of straight lines. The entire period from 1660 to 1920 can be represented within the limits of uncertainty of the observations by five straight lines, the points of discontinuity being at 1784, 1864, 1876, 1897. At these points the mean motion of the moon has apparently changed abruptly. The excesses of the actual mean daily motion over its average value 47435″ are:

1660—1784	$\Delta n'$ =	+ 0″·0007
1784—1864		— ·0005
1864—1876		— ·0021
1876—1897		— ·0011
1897—1917		+ ·0009

TABLE 1. FLUCTUATIONS OF THE MOON'S MEAN LONGITUDE.

t	B′	t	B′	t	B′	t	B′
1621	+ 22″	1785	+ 15″·4	1852·5	+ 3″·50	1900·5	— 15″·53
37	— 16·4	92	+ 14·3	57·5	+ 2·68	03·5	— 14·48
49	— 16·7	1801·5	+ 12·9	62·5	+ 1·41	06·5	— 13·32
62	— 15·8	09·5	+ 12·3	67·5	— 1·77	09·5	— 12·78
81	— 12·1	13·0	+ 11·2	72·5	— 5·94	12·5	— 11·62
1710	— 3·2	21·8	+ 10·1	77·5	— 9·43	15·5	— 10·35
27	+ 2·0	31·5	+ 6·7	82·5	— 10·84	18·5	— 10·20
37	+ 4·9			87·5	— 13·25	21·5	— 10·18
47	+ 8·0	1837·4	+ 4·11	91·5	— 14·47	24·5	— 12·03
55	+ 9·6	43·1	+ 3·81	94·5	— 15·81	25·5	(— 12·39)
71	+ 13·8	48·8	+ 3·42	97·5	— 16·12	26·5	(— 12·35)

After 1917 $\Delta n'$ becomes negative again, but the amount cannot yet be stated.

For a further discussion of these fluctuations see section **7**.

4. *The sun's longitude.*

The observed longitudes of the sun after 1835 were taken from Dr. JONES's paper in *M. N.* **87**, p. 4, whith the only difference that from 1915 the correction of $+ 0^s{\cdot}04$ to the Greenwich observations adopted by JONES has been altered to $+ 0^s{\cdot}03$, according to a private letter from Dr. JACKSON to the writer, represents better the change of personality due to the introduction of the travelling-wire micrometer.*) In accordance with this the systematic correction applied to the Cape series from 1912—1916 was altered to $+ 0''{\cdot}30$. The last two columns of JONES's Table III on p. 9 were altered accordingly, and then means were taken for four, three or two years.

For the period 1750 to 1835 means were formed from the observed errors given in BROWN's paper, Table IV, p. 223, applying a systematic correction of $- 0''{\cdot}66$ to Cambridge, and assigning weights very much on the same principles as was done by JONES in the formation of his Table II.

The observed corrections to NEWCOMB's tables derived in this way are given in Table 2 under the heading $(O-C)_0$. The last two means were not used in the solution, but added afterwards to the table. The last is the mean of the results for 1925 and 1926, which were kindly communicated by the Astronomer Royal.

As a first approximation we adopted the secular acceleration $+ 1''{\cdot}80\,S$ and a correction $+ 2''{\cdot}00\,(T + 1)$ to the mean longitude, and we assumed $Q = 1{\cdot}50$. We thus get

$$(O-C)_1 = (O-C)_0 - [2''{\cdot}00\,(T+1) + 1''{\cdot}80\,S + 0{\cdot}112\,B'],$$

which is also given in the table. It is evident from these that the observations before 1830 are too unreliable. From 1865 to 1895 the residuals $(O-C)_1$ are systematically positive, while from 1835 to 1865 and after 1895 they are small and not strongly systematic. It seems probable that the observed longitude of the sun in the interval 1865—95 requires a negative correction of about $0''{\cdot}60$. It is, of course, also possible that the adopted fluctuations of the moon in this interval require a positive correction of about $5''{\cdot}4$. In both cases it appears better to exclude these years from the final approximation. Accordingly the unknowns a, b, c were determined from the equations of condition

$$(O-C)_1 = a + bT + cB'$$

formed for the normal places from 1839 to 1863 and 1896·5 to 1922. The resulting values were

$$a = -0{\cdot}11$$
$$b = -0{\cdot}59$$
$$c = -0{\cdot}014$$

A slight mistake in the last epoch was discovered afterwards, making the residual for 1922 $-0''{\cdot}24$ as given in the table, instead of $+ 0''{\cdot}01$ as originally found. Correcting this and making a new solution including the last two normal places, we find

$$\delta a = -{\cdot}01 \pm {\cdot}06$$
$$\delta b = -{\cdot}11 \pm {\cdot}25$$
$$\delta c = +{\cdot}004 \pm {\cdot}010.$$

The probable errors have been derived from the residuals. It has not been thought worth while to carry these corrections through, and the original solution has been retained. The correction to NEWCOMB's tables is thus

(5) $\quad \Delta L' = + 1''{\cdot}89 + 1''{\cdot}41\,T + 1''{\cdot}80\,S + 0{\cdot}098\,B'.$

The coefficient of B' corresponds to

(6) $\quad Q = 1{\cdot}31 \pm {\cdot}13.$

The real uncertainty may be larger than is indicated by the probable error, on account of the doubt regarding the systematic corrections of the observations.

The residuals given in Table 2 correspond to the finally adopted value $Q = 1{\cdot}25$, and the adopted values $\Delta L_0' = + 1''{\cdot}89$, $\Delta n_0 = + 1''{\cdot}41$. The last column of the table gives the fluctuations in the sun's longitude multiplied by n/Qn_0 to make them comparable with those of the moon. Thus

(7) $\quad B_0' = 10{\cdot}72\,(O-C)$
$\quad C = \text{NEWCOMB's tables} + 1''{\cdot}89 + 1''{\cdot}41\,T + 1''{\cdot}80\,S.$

The values of B_0' are represented in the figures 3, 4 and 5.

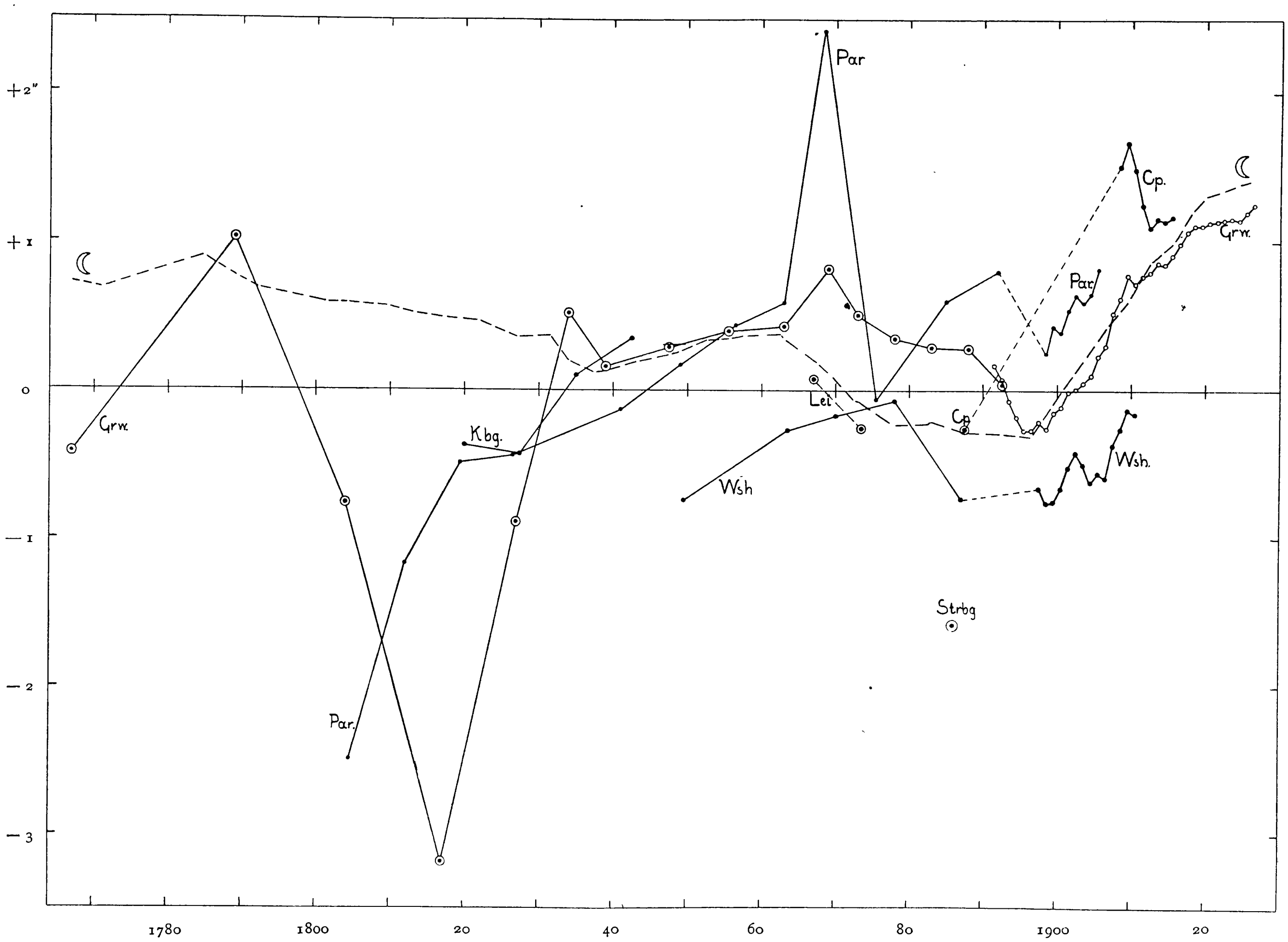
In figure 2 the corrections $(O-C)_0$ to NEWCOMB's tables as derived from the Greenwich observations are represented, the smoothed means for every year being taken after 1890. The figure also contains the results from some other observatories, without any systematic corrections. These were, after 1890, taken from JONES's table III, smoothed by taking the means of every three successive years. The two values for Leiden were taken from *Leiden Annals*, XI, **2**, page B 104. The broken line marked ☾ gives the corrections to NEWCOMB's tables of the sun derived from the observed fluctuations of the moon by the

*) See also *M.N.* 87, p. 459.

TABLE 2. LONGITUDES OF THE SUN.

t	$(O-C)_0$	$(O-C)_1$	Res.	B_0'	t	$(O-C)_0$	$(O-C)_1$	Res.	B_0'
	″	″	″	″		″	″	″	″
1767	— 0·42	— 0·70	— 1·15	+ 0·4	1888	+ 0·28	+ 0·78	+ 0·51	— 7·7
89	+ 1·03	+ ·53	+ ·25	+ 17·5	92·5	+ 0·04	+ ·58	+ ·28	— 11·9
1804	— 0·76	— 1·12	— 1·37	— 1·9	96·5	— 0·27	+ ·14	+ ·02	— 15·7
17	— 3·19	— 3·50	— 3·70	— 29·0	1900·5	— 0·11	+ ·06	— ·09	— 16·5
27	— 0·89	— 1·08	— 1·27	— 5·2	03·5	+ 0·07	+ ·01	— ·14	— 16·0
34	+ 0·52	+ ·39	+ ·30	+ 8·8	06·5	+ 0·34	+ ·02	— ·09	— 14·2
39	+ 0·16	+ ·08	+ ·01	+ 4·2	09·5	+ 0·72	+ ·19	+ ·12	— 11·6
47·5	+ 0·29	+ ·04	+ ·03	+ 3·9	12·5	+ 0·80	— ·02	— ·06	— 12·2
55·5	+ 0·40	+ ·15	+ ·03	+ 3·3	15·5	+ 0·91	— ·17	— ·18	— 12·3
63	+ 0·43	+ ·17	+ ·06	+ 1·7	18·0	+ 1·09	— ·14	— ·11	— 11·5
69	+ 0·82	+ ·83	+ ·68	+ 4·3	20·0	+ 1·12	— ·23	— ·19	— 12·2
73	+ 0·51	+ ·75	+ ·56	— 0·3	22·0	+ 1·14	— ·30	— ·24	— 13·1
78	+ 0·35	+ ·83	+ ·57	— 3·5	24·0	(+ 1·16)	— ·23	— ·19	— 13·9
83	+ 0·29	+ ·70	+ ·49	— 5·9	26·0	(+ 1·22)	— ·23	— ·18	— 14·5

FIGURE 2.



formula (5), replacing however the coefficient of B' 0·098 by 0·094 to conform to the finally adopted value $Q = 1·25$. It will be seen at once from this figure how uncertain the observations before 1830 are. It is earnestly to be hoped that the reduction of the observations made at the Radcliffe Observatory by HORNSBY from 1774 to 1803 and by ROBERTSON and RIGAUD from 1811 to 1838 will help to elucidate this uncertainty.

Also the apparent systematic error of + 0″·60 from 1865 to 1895 is very strikingly shown.

5. *The longitude of Venus.*

The values of Δl *smoothed* given by JONES in his Table VII, *l. c.* p. 19, were plotted and a smooth curve drawn through them, from which the corrections $(O-C)_0$ to NEWCOMB's tables, contained in Table 3, were read off. JONES derived the errors in the longitude of Venus from the observed geocentric longitudes by means of his observed errors of the sun. They are consequently affected by the systematic error of the latter from 1865 to 1895, but the effect on the mean of a few years will not be systematic, since it changes sign at the conjunctions. The observations were treated in the same way as those of the sun, with this difference that a possible correction $\delta\varkappa_2$ to the adopted value $\varkappa_2 = +0{\cdot}80$ was kept explicitly in the right hand members of the equations. The epochs 1840, 1845 and 1850 were not used in the final approximation.

The resulting correction to NEWCOMB's tables is

$$\Delta\lambda_2 = +3''{\cdot}36 + 0''{\cdot}40\,\delta\varkappa_2 + (2''{\cdot}03 - 1''{\cdot}95\,\delta\varkappa_2)\,T + (1''{\cdot}80 + \delta\varkappa_2)\,S_2,$$

where $S_2 = 1{\cdot}63\,S$, and the value found for Q was

$$(8) \qquad Q_2 = 1{\cdot}269 \pm {\cdot}062 + {\cdot}028\,\delta\varkappa_2.$$

The residuals are very small. Table 3 contains the residuals and the value of B_2' corresponding to the finally adopted values $Q = 1{\cdot}25$, $\delta\varkappa_2 = -0{\cdot}25$, viz:

$$B'_2 = 6{\cdot}58\,[(O-C)_0 - \Delta\lambda_2].$$

The effect of the correction $\delta\varkappa_2$ on the residuals is, however, extremely small.

TABLE 3. LONGITUDES OF VENUS.

t	$(O-C)_0$	Res.	B_2'	t	$(O-C)_0$	Res.	B_2'	t	$(O-C)_0$	Res.	B_2'
1840	$-0''{\cdot}30$	$-0''{\cdot}88$	$-1''{\cdot}8$	1875	$+0''{\cdot}50$	$+0''{\cdot}37$	$-5''{\cdot}2$	1904	$+0''{\cdot}55$	$-0''{\cdot}13$	$-15''{\cdot}1$
45	$+0{\cdot}23$	$-{\cdot}46$	$+0{\cdot}7$	80	$+0{\cdot}52$	$+{\cdot}57$	$-6{\cdot}7$	07	$+1{\cdot}02$	$-{\cdot}01$	$-13{\cdot}3$
50	$+0{\cdot}53$	$-{\cdot}26$	$+1{\cdot}5$	84	$-0{\cdot}10$	$-{\cdot}07$	$-12{\cdot}0$	10	$+1{\cdot}31$	$-{\cdot}01$	$-12{\cdot}6$
55	$+0{\cdot}92$	$-{\cdot}04$	$+2.8$	88	$-0{\cdot}17$	$-{\cdot}03$	$-13{\cdot}8$	13	$+1{\cdot}51$	$-{\cdot}18$	$-12{\cdot}6$
60	$+1{\cdot}05$	$+{\cdot}07$	$+2{\cdot}5$	92	$-0{\cdot}18$	$-{\cdot}10$	$-15{\cdot}3$	16	$+1{\cdot}96$	$-{\cdot}08$	$-10{\cdot}9$
65	$+0{\cdot}96$	$+{\cdot}04$	$+0{\cdot}5$	96	$-0{\cdot}14$	$-{\cdot}08$	$-16{\cdot}5$	19	$+2{\cdot}35$	$+{\cdot}08$	$-9{\cdot}7$
70	$+0{\cdot}49$	$+{\cdot}01$	$-3{\cdot}8$	1900	$+0{\cdot}15$	$-{\cdot}08$	$-16{\cdot}2$	22	$+2{\cdot}56$	$+{\cdot}12$	$-9{\cdot}7$

Figure 5 shows the values of B_2' computed for every year from JONES's smoothed means smoothed once more. In the figures 3 and 4 the normal points of Table 3 are represented. It wil be seen that, with the exception of the first years, the agreement of B_2' with the fluctuations of the moon's longitude is remarkably close, in fact the deviations are on the whole not larger than those between the results from the meridian observations of the moon and the occultations.

6. *Transits of Mercury.*

The transits of Mercury have been discussed by NEWCOMB in *Astr. Papers Am. Ephemeris*, Vol. I, p. 363 (1882). A new reduction, including all transits observed since, was given by INNES in *Union Observatory Circular* **65** (1925). INNES gives in Table IX of his paper, p. 321, the observed corrections to the predicted times of the second and third contacts, and in the eleventh column of Table VIII, p. 320, are given the logarithms of the factors by which these must be divided to give the errors in the difference of longitude. In Table IX he further gives the coefficients of the corrections to several elements and to the mass of Venus. I have adopted a correction $\delta m_2 = +0{\cdot}010 m_2$, which is about the mean of the values found by JONES from the sun ($+{\cdot}0080 \pm {\cdot}0019$, *M. N.* **86**, p. 434, 1926) and ROSS from Mars ($+{\cdot}0115 \pm {\cdot}0058$, *Astr. Papers Am. Eph.* **9**, p. 260, 1917). Denoting the coefficient of Table VIII, 11th column, by A, and the coefficient of $10\,\delta m_2/m_2$ (Table IX, 10th column) by e, we have

$$\left.\begin{matrix} V_0 \\ W_0 \end{matrix}\right\} = \Delta v_1 - \Delta v_0 = -\frac{O - C - 0{\cdot}1\,e}{A},$$

Δv_1 and Δv_0 being the corrections to the true longitudes of Mercury and the sun respectively. These were formed for each of the epochs given by INNES, and then the second and third contacts of each transit were combined with the weights given by INNES. The corrections $\Delta v_1 - \Delta v_0$ thus derived are the quantities denoted by V and W by NEWCOMB, *Astr. Papers*, **1**, p. 447, viz:

for November transits: $V = 1.487\,\Delta\lambda_1 - 1{\cdot}01\,\Delta L'$ + corrections to other elements

for May transits: $W = 0{\cdot}716\,\Delta\lambda_1 - 0{\cdot}97\,\Delta L'$ + corrections to other elements.

The coefficients of $\Delta\lambda_1$ and $\Delta L'$ vary a little from one transit to another, but not more than about one percent, and we can safely use the average values throughout.

The weights of the observed corrections are

$$p_1 = 0{\cdot}02\, A^2 p,$$

the factor 0·02 being about the average value of $1/A^2$, and p being proportional to the weight of INNES's $O - C$ (in time). The factor of proportionality was so chosen as to make the probable error of unit weight $\pm$ 1″·0. Half weight was assigned to the transit of May 1924, on account of the unfavourable circumstances of the observations: the second contact shortly before sunset in America, the third and fourth soon after sunrise in Europe and Africa. The values of V_0 and W_0 and their weights p_1 so derived are given in Table 4. It is doubtful whether the weights assigned by INNES to the May transits are not too high. On the whole the residuals given by the May transits are not so much smaller than those of the November transits as to justify these high weights. The weights p of the $O - C$ in time are the same on the average for the May and the November transits. There is no a priori reason, of course, why the observations of the times of contact should be less accurate in May than in November, except that the May phenomena are somewhat slower. Since the May and the November transits were discussed separately, the point is, however, not of much importance, and I have retained the weights as given.

As a first approximation the value of $\Delta L'$ derived above, and an approximation for $\Delta\lambda_1$ found from a preliminary discussion, were introduced, thus

$$\Delta L' = +1''{\cdot}89 + 1''{\cdot}41\, T + 1''{\cdot}80\, S$$
$$\Delta\lambda_1 = +8{\cdot}84 + 8{\cdot}38\, T + 1{\cdot}80\, S_1,$$

where

$$S_1 = 4{\cdot}15\, S.$$

We put

$$Q = 1{\cdot}50\,(1 + \nu) \qquad Q_1 = 1{\cdot}50\,(1 + \nu_1)$$

Then taking as a first approximation $Q = Q_1 = 1{\cdot}40$, or $\nu = \nu_1 = -{\cdot}067$, we have

$$V_1 = V_0 - [1{\cdot}487\, \Delta\lambda_1 - 1{\cdot}01\, \Delta L'] - {\cdot}540\, B'$$
$$W_1 = W_0 - [\,{\cdot}716\, \Delta\lambda_1 - \ {\cdot}97\, \Delta L'] - {\cdot}210\, B'.$$

The coefficients of B' are $0{\cdot}933\,[1{\cdot}487 \times {\cdot}466 - 1{\cdot}01 \times \times {\cdot}112]$ and $0{\cdot}933\,[{\cdot}716 \times {\cdot}466 - {\cdot}97 \times {\cdot}112]$ respectively. The values of V_1 and W_1 are also given in Table 4.

The equations of condition now are

$$V_1 = 1{\cdot}487\,[\partial\lambda_{10} + T\partial n_1 + S_1 \partial \varkappa_1] - 1{\cdot}01\,[\partial L_0' + T\partial n_0 + S\partial\varkappa] + {\cdot}692\, B'\nu - {\cdot}113\, B'\nu_1 + \partial V_0 + T\partial V_1$$
$$W_1 = {\cdot}716\,[\partial\lambda_{10} + T\partial n_1 + S_1\partial\varkappa_1] - {\cdot}97\,[\partial L_0' + T\partial n_0 + S\partial\varkappa] + {\cdot}334\, B'\nu - {\cdot}109\, B'\nu_1 + \partial W_0 + T\partial W_1,$$

where

$$\partial V = \partial V_0 + T\partial V_1$$
$$\partial W = \partial W_0 + T\partial W_1$$

are made up out of corrections to the other elements (e, π, e_1, π_1) and their secular variations. With a view to the remaining uncertainty of these elements as used in NEWCOMB's tables it must be considered quite possible that ∂V and ∂W may be of the order of several tenths of a second of arc. It is, of course, impossible to determine $\partial\lambda_1$, $\partial L'$ and the corrections ∂V and ∂W separately. I have therefore neglected the corrections to the longitude of the sun, thus including their effect in ∂V and ∂W. Even then the only unknowns that can be determined are

$$a_1 + b_1 T = 1{\cdot}487\,(\partial\lambda_{10} + T\partial n_1) + \partial V$$
$$a_1' + b_1' T = \ {\cdot}716\,(\partial\lambda_{10} + T\partial n_1) + \partial W,$$

And the distribution of the corrections found between $\partial\lambda_1$, ∂V and ∂W remains arbitrary.

If we neglect the corrections ∂V and ∂W entirely, and then divide the equations of the November transits by 1·487 and those of the May transits by 0·716, the coefficient of $\partial\lambda_{10} + T\partial n_1$ becomes unity in both cases, and the weights become $(1{\cdot}487)^2 p_1$ and $(0{\cdot}716)^2 p_1$ respectively. Now p_1 contains A^2 as a factor, and the values of A, though varying rather much from one transit to another, especially for those near the northern or southern limbs of the sun, are of the order of about 5 for the November, and 12 for the May transits. Consequently the weights of the equations in which the coefficient of $\partial\lambda_1$ is reduced to unity become on the average equal to the original weights p of the observed errors in time. This is the approximation used above in section **3**. Here, however, I retain the equations without this reduction and with the weights p_1.

We can now either treat the November and the May transits separately, in which case the November transits involve the unknowns

$$a_1,\ b_1,\ \partial\varkappa_1 - {\cdot}164\,\partial\varkappa,\ \partial\nu_1 - {\cdot}164\,\partial\nu,$$

and the May transits

$$a_1',\ b_1',\ \partial\varkappa_1 - {\cdot}326\,\partial\varkappa,\ \partial\nu_1 - {\cdot}326\,\partial\nu,$$

Or we can combine all equations to one set of normals, in which case we must make a hypothesis regarding the corrections ∂V and ∂W, and we must neglect $\partial\varkappa$ and $\partial\nu$. Putting

$$2\partial W - \partial V = x = x_0 + x_1 T$$
$$4\partial W + 3\partial V = y,$$

the corrections $\partial\pi$, ∂e_1, etc., have large coefficients in x and small ones in y. We can thus neglect y, and then we have

$$a_1 + b_1 T = 1{\cdot}487\,(\partial\lambda_{10} + T\partial n_1) - 0{\cdot}4\,(x_0 + x_1 T)$$
$$a_1' + b_1' T = 0{\cdot}716\,(\partial\lambda_{10} + T\partial n_1) + 0{\cdot}3\,(x_0 + x_1 T),$$

and the combined equations contain the unknowns

$$\partial\lambda_{10},\ \partial n_1,\ x_0,\ x_1,\ \partial\varkappa_1,\ \partial\nu_1.$$

The solutions, in which the transit of November 1782 was rejected, gave

November transits	*May transits*	*Combined equations*
$a_1 = -3''.82$	$a_1' = -''.86$	$\delta\lambda_{10} = -1''.99$
$b_1 = -3.75$	$b_1' = -1.12$	$\delta n_1 = -2.11$
$\delta\varkappa_1 - .164\,\delta\varkappa = -.54$	$\delta\varkappa_1 - .326\,\delta\varkappa = -.35$	$\delta\varkappa_1 = -\ .44$
$\delta\nu_1 - .164\,\delta\nu = -.25$	$\delta\nu_1 - .326\,\delta\nu = -.14$	$\delta\nu_1 = -\ .20$
		$x_0 = +\ .72$
		$x_1 = +\ .08$

From a_1, b_1, a_1', b_1' we would find

$$\delta\lambda_{10} = -2.05$$
$$\delta n_1 = -2.15$$
$$x_0 = +2.02$$
$$x_1 = +1.38$$

The values of $\delta\lambda_{10}$ and δn_1 are in excellent agreement with those derived from the combined solution; those of x_0 and x_1 are not, but the weight of their determination is very small.

The determinations of $\delta\varkappa_1$ and $\delta\nu_1$ are unsatisfactory, the unknowns being badly separated. I have therefore made a new solution transferring $\delta\varkappa_1$ to the right hand members. We then find

$$\delta\lambda_1 = -''.76 + 2''.91\,\delta\varkappa_1 + [-1''.20 + 2''.20\,\delta\varkappa_1]\,T + S_1\,\delta\varkappa_1$$
$$\delta V = -.36 - 2.66\,\delta\varkappa_1 + [+\ .11 - 1.96\,\delta\varkappa_1]\,T$$
$$\delta W = +.24 + 1.54\,\delta\varkappa_1 + [-\ .08 +\ .08\,\delta\varkappa_1]\,T$$
$$\delta\nu_1 = -.09 + .20\,\delta\varkappa_1$$

Residuals were computed for the three hypotheses

$$\delta\varkappa_1 = \quad 0\,, \qquad -.40\,, \qquad -.80.$$

The sums of the squares of the residuals, each multiplied by its own weight, were for the three cases

November	44.6	30.8	71.2
May	45.6	71.0	222.2
Sum	90.2	101.8	293.2

Representing these by a parabolic formula, we can interpolate the value of $\delta\varkappa_1$ which makes them a minimum. We find

November	$\delta\varkappa_1 =$	-0.30
May		$-\ .12$
Combined		$-\ .17$

The direct determinations of $\delta\varkappa_1$ from the normal equations gave about

$$\delta\varkappa_1 = -0.40$$

On the whole it seems safe to adopt

$$\delta\varkappa_1 = -0.25 \pm .10$$

The probable error is only a rough estimate, but I expect it to be rather too high than too low.

The corresponding value of $\delta\nu_1$ is -0.14, giving $\nu_1 = -.207$, and

$$(9) \qquad Q_1 = 1.19 \pm .075$$

The probable error corresponds to the weight derived from the combined equations, taking the probable error for unit weight to be $\pm 1''.0$.

Collecting the different determinations of Q, Q_1 and Q_2 given by (3), (3)', (6), (8) and (9) we have

from the sun		$Q =$	$1.31 \pm .13$
from Venus		$Q_2 =$	$1.26 \pm .06$
from Mercury	'great empirical term'	$Q_1 =$	$1.32 \pm .07$
	'minor fluctuations'		$1.34 \pm .02$
	total fluctuations		$1.19 \pm .08$

The difference between the value of Q_1 derived here, and those found in art. **3** is due chiefly to the different treatment of the weights. The comparison of these different determinations, which are independent of each other, makes it very difficult to avoid the conclusion that the true value of Q, Q_1 and Q_2 is the same. Assuming this, we can take as a general mean

$$Q = 1.25 \pm .02$$

The probable error corresponds to the deviation of the different determinations, but it seems likely that the true uncertainty is much larger. I estimate it to be about $\pm .08$.

Adopting now $\delta\varkappa_1 = -0.25$, $Q_1 = Q = 1.25$, a new determination was made of a_1, b_1, a_1', b_1', from which the final corrections to the mean longitude of Mercury and the corrections δV and δW are found to be:

$$(10) \qquad \begin{aligned} \delta\lambda_1 &= +7''.65 + 7''.13\ T + 1''.55\ S_1 + .388\ B' \\ \delta V &= -''.55 - ''.42\ T \\ \delta W &= +\ .53 + .44\ T \end{aligned}$$

We have made δV and δW about equal and of opposite signs, but the three corrections are not independent. We can add an arbitrary quantity z to $\delta\lambda_1$, if we subtract at the same time $1.487\,z$ from δV and $0.716\,z$ from δW.

With the residuals from this final solution two further experimental solutions were made. In one of these the unknowns a_1, b_1, a_1', b_1', $\delta_1\varkappa_1$ were introduced. This gave

$$\delta_1\varkappa_1 = -.03 \pm .07$$

In the other solution, taking the unknowns a_1, b_1, a_1', b_1', $\delta_1\nu_1$, we found

$$\delta_1\nu_1 = -.07 \pm .08$$

These corrections are smaller than their probable errors, and the solution (10) was accordingly adopted as final.

The residuals of this final solution are given in

Table 4. The probable error of unit weight derived from them is $\pm$ 1″·00, in exact agreement with the assumed value on which the weights were based. It will be seen that most residuals are small. Of the modern observations 1894·9 and 1924·4 are the only ones leaving considerable residuals, especially the latter. Attention has already been called to the unfavourable circonstances under which it was observed. The transit of May 1740, to which INNES assigns the weight zero, rests on only one observation of one contact. The November transit of 1782·9, which also leaves an excessive residual, was well observed. The third contact, which is responsible for the discrepancy, occurred very near sunset in Europe, but near noon in America.

TABLE 4. TRANSITS OF MERCURY.

November transits						May transits					
t	V_0	p_1	V_1	Res.	B_1'	t	W_0	p_1	W_1	Res.	B_1'
	″		″	″	″		″		″	″	″
1677·9	− 2·57	0·1	+ 1·06	+ 0·73	− 11·2						
90·9	− 2·32	0·1	+ ·60	+ ·22	− 8·5	1740·4	− 14·66	0	− 13·65	− 13·66	− 67·3
97·9	− 7·09	0·3	− 2·79	− 3·44	− 15·1	53·4	− 0·78	4	− ·21	− ·30	+ 7·2
1723·9	− 1·10	1·0	+ 1·79	+ 1·05	+ 3·1	86·4	− 0·18	15	− ·77	− ·85	+ 10·6
36·9	− 0·42	1·7	+ 1·55	+ ·81	+ 6·6	99·4	+ 0·32	14	− ·01	− ·22	+ 11·3
43·9	− 0·86	1·9	+ ·35	− ·31	+ 6·4	1832·4	+ 0·42	23	+ ·26	− ·03	+ 6·3
56·9	− 0·11	0·2	+ ·09	− ·45	+ 9·2	45·4	+ 0·56	26	+ ·30	+ ·05	+ 3·9
69·9	− 0·50	0·9	− 1·89	− 2·31	+ 8·8	78·4	− 0·06	25	+ ·39	+ ·09	− 9·5
82·9	+ 10·51	[2·9]	+ 8·23	+ 7·95	+ 31·8	91·4	− 0·05	25	+ ·46	+ ·19	− 14·4
89·9	+ 1·38	1·7	− ·53	− ·98	+ 12·5	1924·4	+ 2·59	14	− ·88	− ·62	− 15·3
1802·9	+ 0·34	0·6	− 1·13	− 1·44	+ 9·9						
22·9	+ 0·76	1·1	− ·73	− ·87	+ 8·1						
48·9	+ 1·35	2·7	− ·36	+ ·43	+ 4·5						
61.9	+ 1·82	2·1	− ·17	+ ·27	+ 2·3						
68·9	− 0·08	3·5	+ ·09	+ ·37	− 3·9						
81·9	− 1·23	4·0	− ·27	+ ·14	− 10·2						
94·9	− 0·47	4·3	+ ·51	+ 1·27	− 13·2						
1907·9	+ 2·10	4·1	− 1·64	− ·14	− 13·3						
14·9	+ 4·14	4·5	− 2·60	− ·66	− 11·9						

The transit was nearly a grazing one, the least distance of centres being not 30″ less than the sun's semi-diameter. The discrepancy must thus be ascribed to the difficult nature of the observation.

The last column of Table 4 gives the fluctuations B_1', computed by:

$$\textit{November}: B_1' = 2{\cdot}07\,[V_0 - 1{\cdot}487\,\delta\lambda_1 + 1{\cdot}01\,\delta L' - \delta V]$$
$$\textit{May}: \qquad 5{\cdot}32\,[W_0 - {\cdot}716\,\delta\lambda_1 + {\cdot}97\,\delta L' - \delta W].$$

These are also represented in the figures 3, 4 and 5.

The attention of observers may be called to the coming transit of November 9 of the present year. The importance of this transit for the problems in hand was already pointed out by NEWCOMB.*)

7. *Theoretical considerations.*

The projections of the angular momentum of the system Earth-Moon on the ecliptic and on two planes rectangular to it and to each other are:

$$\begin{aligned} &C\omega\cos\varepsilon + \mu a^2 n \cos i\,(1-e^2)^{\frac{1}{2}} = c_3 \\ (11)\quad &C\omega\sin\varepsilon\sin\psi + \mu a^2 n \sin i \sin \Omega\,(1-e^2)^{\frac{1}{2}} = c_1 \\ &C\omega\sin\varepsilon\cos\psi + \mu a^2 n \sin i \cos \Omega\,(1-e^2)^{\frac{1}{2}} = c_2. \end{aligned}$$

*) *Astr. Papers*, I, p. 484.

The axial rotation of the moon has been neglected, and also the rotations of the earth constituting the variation of latitude. The orbital motion of the earth has also been disregarded. It is easily shown *) that $(dn_0/n_0)/(dn/n)$ is equal to the ratio of the couples produced by the solar and lunar tides multiplied by $a^2 n/a_0^2 n_0$. This factor is 0·000088. We have put:

C = the moment of inertia of the earth referred to its axis of rotation,
ω = the earth's velocity of rotation,
ε = the inclination of the ecliptic,
ψ = the node of the equator on the ecliptic,
μ = the moon's mass (mass of the earth = 1),
a = » » mean distance,
n = » » mean motion,
e = » » excentricity,
i = the inclination of the moon's orbit on the ecliptic,
Ω = the node » » » » » » » .

The ecliptic is taken as an invariable plane. The three quantities c_1, c_2, c_3 must remain constant.

The energy of the system, also neglecting negligible quantities, is

$$E = \tfrac{1}{2}\,C\omega^2 + \tfrac{1}{2}\,\mu a^2 n^2.$$

*) JEFFREYS, *The Earth*, p. 213.

Differentiating (11) and eliminating $d\varepsilon$ and $d\psi$ by forming $dc_3 - \tan\varepsilon\,(\sin\psi dc_1 + \cos\psi dc_2)$, omitting terms containing $\sin(\Omega - \psi)$ or $\cos(\Omega - \psi)$ as a factor, since these will give zero when integrated over a complete revolution of the moon's node (19 years), using KEPLER's third law $a^3 n^2 =$ const. *), and putting in the resulting equation $\cos i = 1$, $(1-e^2)^{\frac{1}{2}} = 1$, we find

$$(12)\qquad \sec\varepsilon\,\frac{d(C\omega)}{C\omega} - k\frac{dn}{n} - 3k\,(e\,de + \sin i\,di) = 0,$$

where

$$k = \frac{\mu a^2 n}{3\,C\omega} = 1{\cdot}62.$$

If we neglect the variation of e and i, (12) becomes

$$(12a)\qquad \sec\varepsilon\,\frac{d(C\omega)}{C\omega} - k\frac{dn}{n} = 0.$$

If instead of eliminating $d\varepsilon$ and $d\psi$ we had neglected them, and then eliminated di and $d\Omega$ and neglected de, we would have found

$$(12b)\qquad \cos\varepsilon\,\frac{d(C\omega)}{C\omega} - k\frac{dn}{n} = 0,$$

which is also found from $dc_3 = 0$ neglecting $d\varepsilon$, de and di.

The equation that is usually taken is neither (12a) nor (12b), but

$$(12c)\qquad \frac{d(C\omega)}{C\omega} - k\frac{dn}{n} = 0.$$

Although the variations $d\varepsilon$, de and di are actually absolutely negligible, nevertheless the equations (12a) and (12b) derived by neglecting them, as well as (12c), are, of course, inaccurate, and we must use (12).

In *B.A.N.* **117** (12b) was used, and the ratio $Q_0 = 2{\cdot}30$ derived from it, giving $5''{\cdot}92\,T^2$ in the moon's longitude for $1''{\cdot}00\,T^2$ in the sun, was taken as the basis of the first approximation. It disappears, however, entirely from the final results, which are thus independent of the choice of any of the equations (12a), (12b) or (12c), and in fact of any theoretical considerations.

In the equation (12) we put

$$3\,(e\,de + \sin i\,di) = -f\frac{dn}{n}.$$

It is not possible to determine de, di and $d\varepsilon$ separately. They can, however, be determined from (10) in terms of f. We find, taking the secular values only,

$$(13)\qquad \begin{aligned} d\varepsilon &= -k\,(1-f)\,\sin\varepsilon.\,\frac{dn}{n} \\ di &= \frac{1}{3}\sin i\,(1-f).\,\frac{dn}{n} \\ e\,de &= -\frac{1}{3}\,(\sin^2 i + f\cos^2 i)\,\frac{dn}{n}. \end{aligned}$$

The variations $d\psi$ and $d\Omega$ are purely periodic.

As to the first term of (12) $d(C\omega)/C\omega$, we must keep in mind that what we actually observe is not the rotation of the earth as a whole, but of its outer crust, or even, for the greater part of our material, of the Greenwich Observatory. To allow for the possibility of a difference in the rate of change of the rotation of this and the whole earth, I will suppose the earth to consist of two parts, of which the moments of inertia are $C(1-p)$ and Cp, the variations of their rotations being $d\omega$ and $d\omega'$ respectively. Then

$$\frac{d(C\omega)}{C\omega} = \frac{dC}{C} + \frac{(1-p)\,d\omega + p\,d\omega'}{\omega} = \frac{dC}{C} + (1+\Theta)\frac{d\omega'}{\omega},$$

$d\omega'$ being the observed change of rotation. The factor Θ thus allows for a possible difference of this from that of the earth as a whole. Θ may be either positive or negative, but it is probably small.

The observed change of the moon's apparent mean motion, i. e. the mean motion referred to astronomical time, is given by

$$\frac{dn'}{n} = \frac{dn}{n} - \frac{d\omega'}{\omega}.$$

If we put

$$\frac{d\omega'}{\omega} = -Q\,\frac{dn'}{n},$$

we have

$$\frac{dn}{n} = (1-Q)\,\frac{dn'}{n},$$

and consequently (12) becomes

$$(14)\qquad \frac{dC}{C} = Q\,(1+\Theta)\frac{dn'}{n} - (Q-1)\,(1-f)\,k\cos\varepsilon\,\frac{dn'}{n}.$$

The change of energy of the system is

$$\frac{dE}{C\omega^2} = \frac{1}{2}\frac{dC}{C} + (1+\Theta)\frac{d\omega'}{\omega} - \frac{kn}{\omega}\frac{dn}{n},$$

or by a similar reduction, and using (14)

$$(15)\qquad \frac{dE}{C\omega^2} = -\frac{1}{2}\,Q\,(1+\Theta)\frac{dn'}{n} - \frac{1}{2}\,(Q-1)\,k\left[(1-f)\cos\varepsilon - \frac{2n}{\omega}\right]\frac{dn'}{n}.$$

---

*) Strictly speaking the 'constant' is not constant, but $a^3n^2 = fM(1+\mu)\left(1+\frac{Jb^2}{a^2}\right)$, with $Jb^2 = \frac{3}{2}\frac{C-A}{M}$, and consequently it is affected by changes of C. The effect on n is, however, of the order of 1/4000 of that produced through (12).

Introducing numerical values (14) and (15) become

$$(14')\quad \frac{dC}{C} = (1+\Theta)\frac{dn'}{n} - (Q-1)\,[0\cdot49 - 1\cdot49 f - \Theta]\,\frac{dn'}{n}$$

$$(15')\quad \frac{dE}{C\omega^2} = -\frac{1}{2}(1+\Theta)\frac{dn'}{n} -$$

$$-\frac{1}{2}(Q-1)\,[2\cdot45 - 1\cdot49 f + \Theta]\,\frac{dn'}{n},$$

where the effect of a deviation of Q from unity has been brought into evidence.

In the case of the secular acceleration, if it is ascribed to tidal friction, the moment of inertia is not affected. Consequently, putting $dC = 0$ in (14′), we must have

$$(0\cdot49 - 1\cdot49 f)\,(Q-1) = 1.$$

We have taken $\Theta = 0$, since it is inconceivable that the rotation of the crust, or of any part of it, should be secularly different from that of the whole earth.

If we take $d\omega'/\omega = -dn_0/n_0$, the observations give, adopting $\varkappa = +0\cdot80$,

$$Q_s = 4\cdot6.$$

With $f = 0$ we would have $0\cdot49\,(Q-1) = 1$, or *)

$$Q_s = 3\cdot05.$$

The difference between these two values constitutes the apparent contradiction which was referred to in art. 2. There is, however, no a priori reason to regard the distribution of the perturbations due to tidal friction over the three elements corresponding to $f = 0$ as more probable than any other. From $Q = 4\cdot6$ we find **)

$$f = +0\cdot14.$$

This by (13) and with $\frac{dn}{n} = -3\cdot6\,\frac{dn'}{n}$ gives

$$de = +3\cdot13\;\frac{dn'}{n} = +0''\cdot0039\;T$$

$$di = -0\cdot093\;\frac{dn'}{n} = -0''\cdot00012\;T$$

$$d\varepsilon = +2\cdot00\;\frac{dn'}{n} = +0''\cdot0025\;T$$

These secular variations are, of course, too small to be detected by observations.

DARWIN has computed the effect of the friction of bodily tides in a viscous earth on the excentricity and the inclinations of the moon's orbit and the equator on the invariable plane. He finds variations of the same sign, and of nearly the same mutual proportion, as those found here, but much larger. ***)

Of course these results for a planet of small viscosity cannot be transferred to the actual earth, on which the tidal friction and its reaction on the moon is most probably not produced by the bodily tides, nor even by the ocean tides, but, as explained by JEFFREYS, by the skin friction in regions of strong tidal currents, and the secondary tidal waves set up by the reaction of these on the ocean *). Still they may be taken as an indication that a small positive value of f is not improbable.

Taking $\Theta = 0$, $Q = 4\cdot6$, $f = +0\cdot14$, we find from (15′).

$$\frac{dE}{C\omega^2} = -4\cdot53\;\frac{dn'}{n} = -2\cdot74\,.\,10^{-8}\text{ per century.}$$

JEFFREYS finds from the sum of all shallow seas investigated by him **)

$$dE = -1\cdot1\,.\,10^{19}\text{ ergs per second,}$$

or

$$\frac{dE}{C\omega^2} = -0\cdot9\,.\,10^{-8}\text{ per century.}$$

Thus, in order to explain the observed secular accelerations of the sun and the moon, we must suppose that the seas not investigated by JEFFREYS contribute about twice as much as those investigated. This does not appear improbable, seeing that the Behring Sea alone is responsible for more than two thirds of the total found, and considering that the polar seas, where perhaps the friction may be increased by the action of the ice, were not included in his investigation.

Consider now the equations (14) and (15) or (14′) and (15′) for the fluctuations. We have seen (page 25) that these are equivalent to a series of changes in the moon's mean motion, taking place suddenly, or at least within a few years, which are of the same order of magnitude as those produced by the secular acceleration in the course of a century. Taking the observed value of Q, viz: $Q = 1\cdot25$, and the value of f derived from the secular acceleration, viz: $f = +0\cdot14$, then from

$$\frac{\delta n'}{n} = +4.10^{-8},$$

which is about the change in 1897, we find for three different values of Θ

$\Theta =$	$-0\cdot20$		0		$+0\cdot20$	
$10^8\,\frac{\delta C}{C} =$	$+3\cdot2$	$-0\cdot5$	$+4\cdot0$	$-0\cdot3$	$+4\cdot8$	$-0\cdot1$
$10^8\,\frac{\delta E}{C\omega^2} =$	$-1\cdot6$	$-1\cdot0$	$-2\cdot0$	$-1\cdot1$	$-2\cdot4$	$-1\cdot2$

*) From the equations (12 *b*) and (12 *c*) we would find $Q_s = 2\cdot30$ and $Q_s = 2\cdot66$, corresponding to $f = -\cdot18$ and $f = -\cdot09$ respectively.

**) From $Q_s = 4\cdot4$ (see p. 36), we would find $f = +0\cdot13$.

***) *Collected Works*, Vol II, fifth paper, especially pages 329 and 349. See also the pages 381 and 382.

*) JEFFREYS, *The Earth*, p. 210.

**) JEFFREYS, *The Earth*, p. 220.

The parts corresponding to $Q = 1$ and to the excess of Q over unity have been kept separate. It will be seen that the increase of the moment of inertia is very little affected by the increase of Q *). It is only for much larger values of Q, approaching the value for the secular acceleration, that δC becomes small. The dissipation of energy corresponding to the increase of Q from 1 to 1·25, on the other hand, is considerable. It is practically independent of Θ, while the dissipation for $Q = 1$ increases with Θ.

The hypothesis $Q = 1$ would mean that the moon's true mean motion does not change ($dn = 0$). We can then consider the change of the moment of inertia as the primary cause of the fluctuations. This produces the dissipation (or generation as the case may be) of energy corresponding to the first terms given above. For $\Theta = 0$ and for $dn'/n = +4.10^{-8}$ this amounts to $\delta E = -8.10^{28}$ ergs. The changes of C are very great. In order to get an idea of their order of magnitude we may for a moment suppose them to be produced by local displacement of masses. The displacement of a mass μ from the distance r from the axis of rotation to $r + \delta r$ would give

$$\frac{\delta C}{C} = 6\,\frac{\mu}{M}\cos^2\varphi\,\frac{\delta r}{r},$$

M being the mass of the earth and φ the latitude. The effect of a displacement of the whole of the central Asian highlands, including the Himalaya and the Kven Lin, over its own height would produce a change of the order of $\delta C/C = 10^{-8}$, i. e. about one fourth of the change in 1897. It is, of course, rather improbable that catastrophes of this order of magnitude should have happened in historical times without producing any other effects that would have been noticed by geologists. It is also evident that the displacement of mass produced even by many thousands of earthquakes is entirely inadequate. If the displacements are not local, but distributed throughout the body of the earth, or a considerable part of it, they need only be comparatively small, as BROWN has pointed out **). They would correspond to the expansion or contraction produced by a change of temperature of a fraction of a degree. But the work done against (or by, as the case may be) gravitation would still be enormous. We are compelled, as BROWN has convincingly shown, to ascribe the changes of C to some deep seated origin, however difficult it may be to imagine a cause which can produce such enormous effects in so short a time.

The dissipation of energy needed to explain the excess of Q over unity (which is accompanied by a comparatively small change in C) is in our example 4.10^{28} ergs, i. e. of the order of 10^9 times the dissipation by tidal friction in one second. This must be produced within an interval of the order of few years, say 10^8 seconds at most. It must be due to a force finding its origin in the moon, like tidal friction. This force should thus be about ten times as powerful as that acting in tidal friction. In our example, which was about realised in 1897, it is of the same sign as tidal friction, *i. e.* the rotation of the earth is retarded. But there must on the average be as many occasions when it is negative as when it is positive. In 1864 there occurred a shortening of the day of about three quarters of the lengthening in 1897. Now friction can never generate energy, and an interaction between the motion of the moon and the rotation of the earth by any other mechanism than tidal friction is hardly conceivable. It would thus appear that the excess of Q over unity cannot be explained in this way, which is also the point of view taken by BROWN.

We must remember that the factor Q is primarily not more than a mathematical symbol used to describe the observed facts. Our discussion shows that it is very difficult to ascribe a physical meaning to any other values of Q than $Q = 1$ or $Q = Q_s$, the value for the secular acceleration. Now it may well be that the tidal friction, and the secular accelerations produced by it, vary irregularly, the values found in art. **2** being the average over the last 2000 or 2500 years. Supposing then the fluctuations in the longitudes of the moon, the sun and the planets to be the combined effect of this variability of tidal friction, and of the sudden changes of the moment of inertia considered just now, then for the part of the fluctuations produced by this latter cause the ratio of the change in the rotation of the earth and in the moon's mean motion will be given by $d\omega'/\omega = -dn'/n$, and for the former $d\omega'/\omega = -Q_s\,dn'/n$. If the ratio of the action produced by the two causes be p/q, we must have

$$p + q = 1$$
$$p + Q_s q = Q.$$

Taking

$$Q_s = 4{\cdot}6, \qquad Q = 1{\cdot}25$$

we find *)

$$p = 0{\cdot}93, \qquad q = 0{\cdot}07$$

The first cause (change of the moment of inertia) corresponds to a representation of the fluctuations by a series of straight lines. The second (variability of secular acceleration), is equivalent to a representation by a series of parabolas. These represent a term in T^2, which is necessarily negative in some

*) The part due to $Q - 1$ would become zero for $\Theta = +0{\cdot}28$.

**) *Yale Observatory Transactions*, Vol. III, Part 6, p. 232.

*) With $Q_s = 4{\cdot}4$ we would find $p = 0.926$, $q = 0{\cdot}074$.

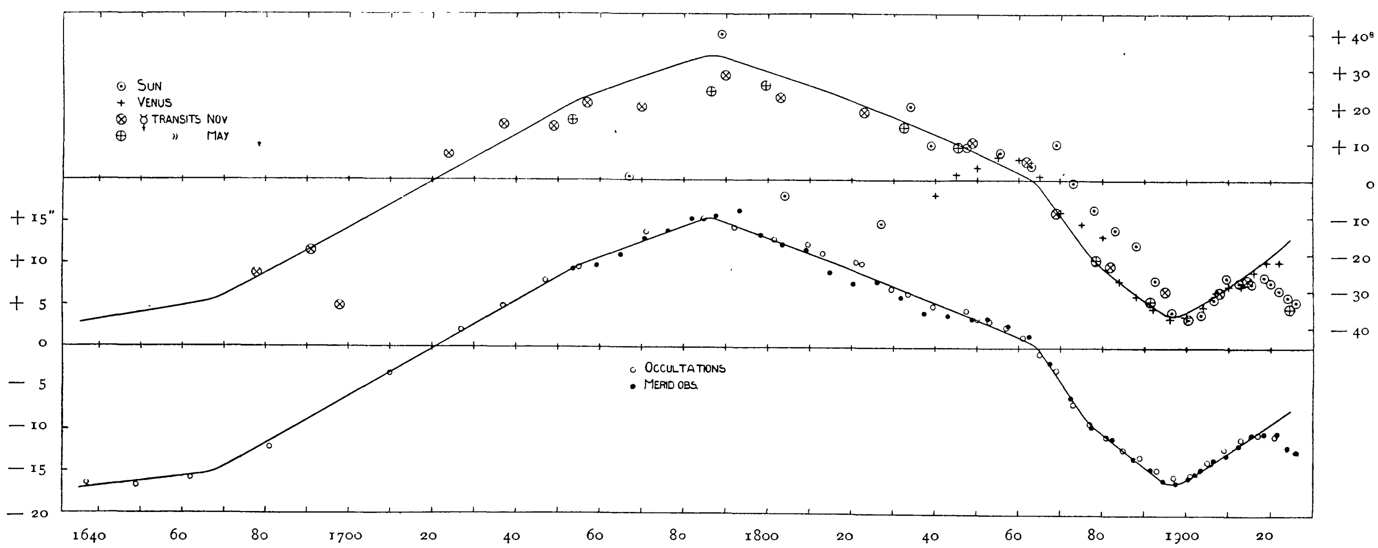
cases. We are thus limited in the choice of our parabolas, since the total coefficient of T^2 must be positive. We take three parabolas, corresponding to three values of the secular acceleration, as follows:

Interval	Coefficient of T^2 in moon's longitude	$10^8 \frac{dE}{C\omega^2\, dT}$ per century	$10^{-19} \frac{dE}{dt}$ ergs per second	Coefficient of T^2 in sun's longitude
1630—1742	+ 5″·64	— 2·9	— 3·6	+ 1″·94
1742—1880	+ 3·17	— 1·7	— 2·2	+ 1·10
1880—1917	+ 15·52	— 8·1	— 10·1	+ 5·35
average	+ 5·22	— 2·7	— 3.4	+ 1·80

The dissipation of energy needed for the steepest interval (1880—1917) is about 9 times that computed by JEFFREYS from the shallow seas, and 3 times that corresponding to the average of the last 2000 years, which is repeated in the last line. This interval is much shorter than the other two, no representation after 1917 being attempted. It is quite possible, and in fact the run of the residuals after 1920 make it rather probable, that from a larger interval we would have found a smaller coefficient.

For the straight lines representing the first cause (change of moment of inertia) we adopt the following excesses of the moon's mean daily motion and of

FIGURE 3.



the length of the day over their average values (47435″ and 86400s respectively):

Interval	$\Delta n'$	$\Delta\tau$	Discontinuities $10^8 \frac{\delta C}{C}$	$10^{-28} \delta E$ ergs
1630—1667	+ 0″·00013	+ 0s·00023	+ 1·3	— 2·5
1667—1758	+ ·00073	+ .00134	— 0·7	+ 1·5
1758—1784	+ ·00041	+ ·00075	— 1·8	+ 3·7
1784—1864	— .00050	— ·00091	— 3·1	+ 6·1
1864—1876	— ·00196	— ·00357	+ 2·0	— 3·9
1876—1897	— ·00102	— ·00186	+ 3.9	— 7.8
1897—1917	+ ·00084	+ ·00153		

These are equal to 0·93 times those given on page 25, with a slight refinement before 1780, which is not really necessary for the representation of the observations, having regard to their uncertainty, but diminishes somewhat the jumps at the epochs of discontinuity.

The changes of the moment of inertia, and the dissipation or generation of energy involved, at the points of discontinuity have been added.

Recapitulating we represent the fluctuations in the longitudes of the moon, the sun and the planets by the combination of: (A) a series of discontinuous changes in the rate of rotation of the earth, produced by changes of the moment of inertia, and (B) a series of changes in the coefficient of tidal friction. The fluctuations are then

in the moon's longitude : $0·93\,(A) + 0·07\,(B)$

in astronomical time and the longitudes of the sun and planets : $0·93\,(A) + 0.32\,(B)$

The two causes act independently of each other, the epochs of discontinuity being different.

In figure 3 the lower curve is $0·93\,(A) + 0·07\,(B)$ and the observed fluctuations of the moon are plotted against it. The upper curve is $0·8\,[0·93\,(A) + 0·32\,(B)]$ and the observations of the sun, Venus and the transits of Mercury are plotted against it. The factor

0·8 reduces the two curves to the same scale, and they are hardly distinguishable on the scale of the figure. To the left is given a scale of seconds of arc of the moon's longitude for the lower curve, and to the right a scale of seconds of time for the upper curve.

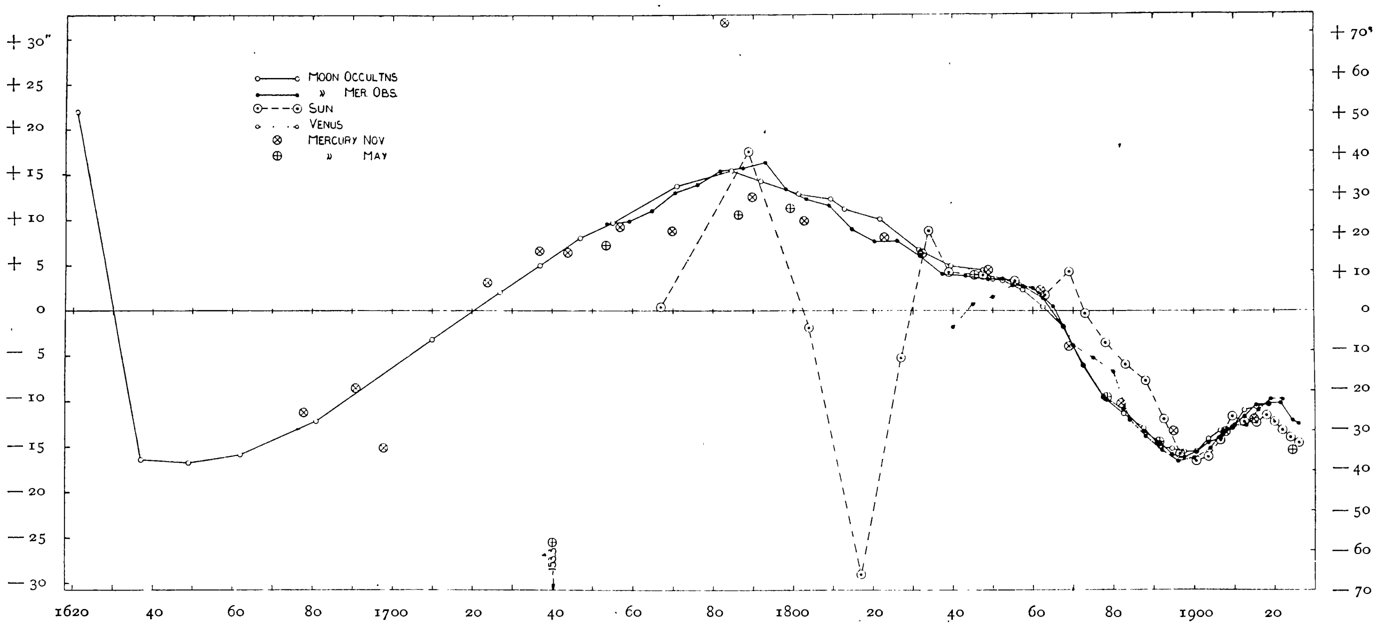
8. *Summary and conclusion.*

The observed longitudes of the sun and Venus, and the observations of the transits of Mercury, were compared with the observed longitudes of the moon. The corrections found to BROWN's tables of the moon, after removing the empirical sine term $10''\cdot71 \sin (140^\circ\cdot0\, T + 240^\circ\cdot7)$, and to NEWCOMB's tables of the sun, Mercury and Venus, apart from the irregular fluctuations, are

$$\begin{aligned}
\Delta L &= +6''\cdot70 + 4''\cdot00\; T \quad + 5''\cdot22\, S \\
&= -0\cdot38 + 0\cdot50\,(T+1) + 5\cdot22\,(T+1)^2 \\
\Delta L' &= +1''\cdot89 + 1''\cdot41\; T \quad + 1''\cdot80\, S \\
&= -0\cdot58 + 0\cdot20\,(T+1) + 1\cdot80\,(S+1)^2 \\
\Delta\lambda_1 &= +7''\cdot65 + 7''\cdot13\; T \quad + 6''\cdot43\, S \\
&= -3\cdot27 + 2\cdot82\,(T+1) + 6\cdot43\,(T+1)^2 \\
\Delta\lambda_2 &= +3''\cdot26 + 2''\cdot52\; T \quad + 2''\cdot53\, S \\
&= -0\cdot75 + 0\cdot82\,(T+1) + 2\cdot53\,(T+1)^2.
\end{aligned}$$

The large constant terms and factors of T are for

FIGURE 4.



the greater part due to the introduction of the secular accelerations. The expressions in terms of $T+1$ show that the tables require but small corrections at the epoch 1800.

The adopted secular accelerations correspond to

$$\begin{aligned}
\varkappa &= +0\cdot80 \pm \cdot16 \\
\varkappa' &= -0\cdot12 \pm \cdot05 \\
\varkappa_1 &= +0\cdot55 \pm \cdot10
\end{aligned}$$

For Venus no independent determination of the secular acceleration was made. The value of $\Delta\lambda_2$ stated above assumes $\varkappa_2 = \varkappa_1$. If we take $\varkappa_2 = \varkappa$ it becomes

$$\begin{aligned}
\Delta\lambda_2 &= +3''\cdot36 + 2''\cdot03\; T \quad + 2''\cdot93\, S \\
&= -0\cdot40 + 0\cdot07\,(T+1) + 2\cdot93\,(T+1)^2.
\end{aligned}$$

The representation of the observations remains quite as good, except for the first three normal epochs.

If the secular accelerations are ascribed to a retardation of the rotation of the earth by tidal friction, then $\varkappa$ and $\varkappa_1$ should be the same. The probable errors, which I judge to be a true measure of the reliability of the determination, do not exclude the equality. We would then be led to adopt *)

$$\varkappa = \varkappa_1 = +0\cdot65.$$

The corresponding terms in the longitudes then become

$$+1''\cdot65\, S, \qquad +6''\cdot95\, S, \qquad +2''\cdot69\, S,$$

for the sun, Mercury and Venus respectively.

*) Taking $\varkappa = +0\cdot65$, we find from the equations of condition of page 23 $\varkappa' = -0\cdot15$, and the coefficient of S in the moon's longitude becomes $5''\cdot03$. The residuals of these equations then become $+\cdot06$, $+\cdot05$, $+\cdot09$, $-\cdot34$, $\cdot00$. The ratio Q_s becomes $4\cdot37$.

The ratio of the secular accelerations of the sun and the moon is $Q_s = (dn_0/n_0)/(dn'/n)$. Adopting $\varkappa = +0{\cdot}80$ we have

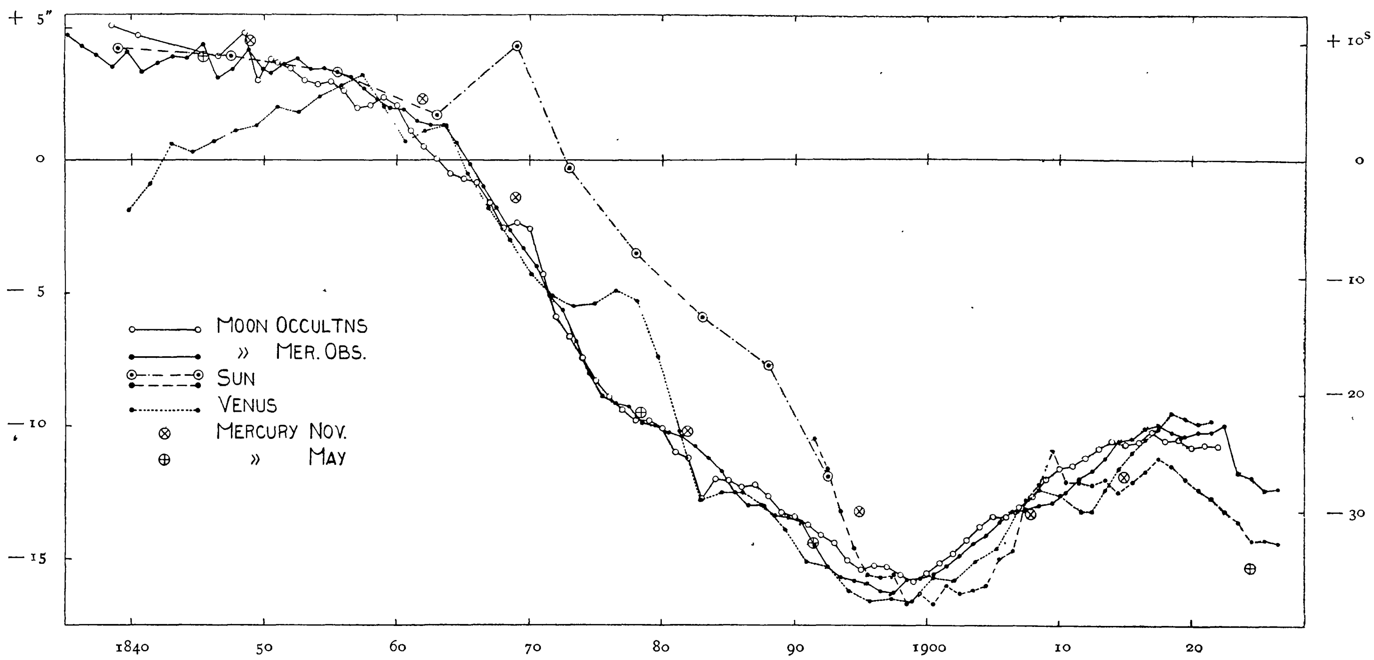
$$Q_s = 4.6.$$

For $\varkappa = +0{\cdot}65$ we would have $Q_s = 4{\cdot}4$

The value of this ratio derived from the observations is much larger than would be expected from the theory of tidal friction as it is usually presented. It has been shown, however, that the usual treatment is incomplete, in so far as it neglects the change in any elements of the moon's motion other than the mean distance. If the complete equations are used, the moon's orbital momentum is found to be decreased by an increase of the excentricity. The compensation by the increase of the mean distance must thus be larger than with the usual formula, and the ratio Q_s is increased. The secular variations of the excentricity and inclination of the moon's orbit and the inclination of the ecliptic needed to account for the observed ratio are far below the limits of possible detection by observations.

After the application of the above corrections, there remain in the longitudes of the moon, the sun, Mercury and Venus irregular fluctuations. A special investigation by a comparison of the transits of Mercury with the moon's longitude showed (section 3) that the distinction, which has often been made, between the 'great empirical term' and the 'minor fluctuations' has no real basis. The fluctuations are much better represented by a series of straight lines. The adopted fluctuations of the moon's longitude are given in Table 1, page 25.

FIGURE 5.



It was found that the fluctuations of the other bodies are proportional to those of the moon, the ratio being

$$1{\cdot}25\ n_i/n.$$

The factor $Q = 1{\cdot}25$ was found independently from the sun, from Venus and from the transits of Mercury, the interagreement of the different determinations corresponding to a probable error of $\pm\ {\cdot}02$ in the final value of Q. Owing on the one hand to the remaining uncertainty regarding the systematic errors of the observations, especially of the sun, and on the other hand to the interdependence of the factor Q and the secular acceleration, the true probable error must be much larger. Nevertheless I consider the excess of Q over unity as well established. From the material discussed in this paper I would judge the true probable error to be about $\pm\ {\cdot}08$, so that *e.g.* the chance of the true value being inside the limits $0{\cdot}95$ and $1{\cdot}05$ would be about $1/25$.

The observational evidence is brought together in figure 4. The scale of seconds of arc at the left hand side of the figure refers to the moon's longitude. The scale of time at the right hand side gives the corrections to the time corresponding to the observed corrections to the longitudes, multiplied by 1·25 in the case of the moon. Figure 5 gives the fluctuations after 1835 on an enlarged scale.

The large value of Q found from Jupiter's satellites in *B. A. N.* **117**, viz.: $Q = 2{\cdot}62 \pm 0{\cdot}20 - 0{\cdot}55\,\varkappa$, thus becomes well nigh impossible. With $\varkappa = +0{\cdot}80$ it would become $Q = 2{\cdot}18$ which is still entirely inadmissible. The determination from the satellites rests on the epochs 1750, 1783, 1891, 1902—3 and 1913—24. The first of these is the epoch of DAMOISEAU's tables, which were based on observations of eclipses, the second depends on a new reduction of eclipses of Satellite III. The epochs 1891 and 1901—2 depend on heliometer observations and 1913—24 on photographic plates. If the different kinds of observations were not comparable, the evidence for the large value of Q would loose its force, for it may well be possible to represent the interval 1891—1924 by $Q = 1{\cdot}25$. We may, of course, always have recourse to the explanation, which, however, should only be invoked in the extreme necessity, that the system of Jupiter has its own fluctuations, independent of those in the moon and the planets or of the rotation of the earth. I will not discuss this point further at the present moment.

The observed longitudes of the moon, the sun and Venus and the transits of Mercury have been discussed apart from any reference to an explanation. The striking parallelism between the fluctuations of the different bodies, and the equality of the factor Q derived independently from the sun and the two planets, make it very difficult to escape the conclusion that the origin of the fluctuations, as well as of the secular acceleration, is in the rotation of the earth. In art. 7 we have seen that all observed facts can be satisfactorily explained by the hypothesis that the actual fluctuations arise from the superposition of the effects of two causes. The first of these is a series of abrupt changes in the rate of rotation of the earth caused by changes of the moment of inertia due perhaps to expansions and contractions of the earth, and the other a variability of the coefficient of tidal friction. The first cause corresponds to the factor $Q = 1$, the other to $Q = Q_s$. The combination of the two causes gives rise to an apparent factor $Q = 1{\cdot}25$. The representation of the observations by this hypothesis is shown in figure 3.

If we accept this hypothesis, then the 'astronomical time', given by the earth's rotation, and used in all practical astronomical computations, differs from the 'uniform' or 'Newtonian' time, which is defined as the independent variable of the equations of celestial mechanics. The correction to be applied to astronomical time in order to get uniform time consists of two parts. One is a secular term

$$\Delta_2 t = +43^{s}{\cdot}8\ S,$$

where $S = T^2 + 1{\cdot}33\,T - 0{\cdot}26$. The coefficient corresponds to the adopted value $\varkappa = +0{\cdot}80$. If we adopt $\varkappa = +0{\cdot}65$ it becomes $40^{s}{\cdot}2$.

In addition to this there are irregular corrections, which for the period 1640 to 1926.5 are given in the following table. From 1640 to 1917 the representation of art. 7 has been adopted, after 1917 the Greenwich meridian observations of the moon were used.

TABLE 5. CORRECTIONS FROM ASTRONOMICAL TO UNIFORMLY ACCELERATED TIME.

t	$\Delta_1 t$	t	$\Delta_1 t$	t	$\Delta_1 t$	t	$\Delta_1 t$	t	$\Delta_1 t$	t	$\Delta_1 t$
	s		s		s		s		s		s
1640	− 38·5	1720	− 0·6	1800	+ 29·7	1840	+ 12·4	1880	− 23·7	1918	− 23·0
50	− 36·7	30	+ 5·8	05	+ 27·8	45	+ 9·9	85	− 28·4	20	− 23·3
60	− 34·8	40	+ 12·3	10	+ 25·9	50	+ 7·4	90	− 32·7	22	− 23·1
70	− 31·9	50	+ 18·8	15	+ 23·8	55	+ 4·7	95	− 36·0	24	− 26·9
80	− 25·8	60	+ 24·2	20	+ 21·7	60	+ 1·9	1900	− 35·9	25·5	− 28·3
90	− 19·6	70	+ 28·3	25	+ 19·5	65	− 1·8	05	− 32·7	26·5	− 28·2
1700	− 13·4	80	+ 32·0	30	+ 17·2	70	− 9·6	10	− 28·9		
10	− 7·0	90	+ 33·5	35	+ 14·9	75	− 17·6	15	− 24·9		

The total correction from astronomical to uniform time is

$$\Delta t = \Delta_1 t + \Delta_2 t.$$

If we wish to adopt $\varkappa = +0{\cdot}65$, $\varkappa' = -0{\cdot}15$, then the coefficient of S in $\Delta_2 t$ becomes $40^{s}{\cdot}2$ instead of $43^{s}{\cdot}8$, and to the fluctuations B' we must add $0''{\cdot}19\,S$, in consequence of which $0^{s}{\cdot}4\,S$ must be added to the values of $\Delta_1 t$ as given in table 5. From the sum $\Delta t = \Delta_1 t + \Delta_2 t$ we must thus subtract $3^{s}{\cdot}2\,S$.