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COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

The expanding universe. Discussion of Lemaître's solution of the equations of the inertial field, by *W. de Sitter.*

1. *The differential equations.*

In *B. A. N.* 185 it was pointed out that neither of the two possible static solutions of the differential equations

$$(1) \qquad G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G + \lambda g_{\mu\nu} + \varkappa T_{\mu\nu} = 0$$

can represent the observed facts of a finite density of matter in space and a systematic velocity of recession of the extragalactic nebulae proportional to the distance, and mention was made of the non-static solution found by Dr. LEMAÎTRE, which is compatible with these two observed facts. In the present article I will discuss some of the consequences of this solution, and will begin by recapitulating it in a notation slightly different from LEMAÎTRE's own. [1])

The conditions of perfect spherical symmetry (isotropy) and perfect homogeneity require the three-dimensional space to be of constant curvature, the three-dimensional line-element thus being

$$(2) \qquad R^2 d\sigma^2 = R^2 \left[d\chi^2 + \sin^2\chi \left(d\psi^2 + \sin^2\psi \, d\theta^2\right)\right].$$

Further the material energy tensor is assumed to be

$$T_{ij} = -g_{ij} p, \quad T_{i4} = 0, \quad T_{44} = g_{44} \rho,$$

where $\rho = \rho_0 + 3p$ is the "relative" density, ρ_0 being the material, or "invariant" density, and p is the pressure, made up of the "kinematical" pressure corresponding to the random motions of the particles, or "molecules", of which the matter is conceived to consist, and the radiation pressure corresponding to the energy of radiation which may be present.

The four-dimensional line-element then is

$$ds^2 = -R^2 d\sigma^2 + f\, dt^2.$$

In LEMAÎTRE's solution R and f are taken to be functions of t alone. Since we can always put $f dt^2 = c^2 d\tau^2$ and use τ as a new independent variable, we may take $f = c^2$, c being the velocity of light.

The equations (1) then become, if we denote differential quotients d/cdt by dots,

$$(3) \qquad \begin{aligned} 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} &= \lambda - \varkappa p \\ \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} &= \frac{1}{3}(\lambda + \varkappa \rho) \end{aligned}$$

and the equation of energy is

$$(4) \qquad \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = 0.$$

LEMAÎTRE puts

$$(5) \qquad \varkappa\rho_0 = \frac{\alpha}{R^3}, \qquad \varkappa p = \frac{\beta}{R^4}.$$

The equation (4) can then be written

$$(4') \qquad R\dot{\alpha} + 3\dot{\beta} = 0.$$

The three equations (3) and (4), or (3) and (4′) are not independent of each other: (4′) can easily be derived from (3). We will use the second of (3) and (4′). An assumption regarding α, or β, must evidently be added in order to make a complete solution possible. The total volume of space is $\pi^2 R^3$, consequently the total mass is, by (5), $\pi^2 \alpha/\varkappa$. This LEMAÎTRE takes to be constant, and consequently by (4′) β is also constant. LEMAÎTRE takes $\beta = 0$.

If we put

$$(6) \qquad y = R\sqrt{\lambda}, \qquad \text{A} + \delta = \alpha . \sqrt{\lambda}, \qquad \varepsilon = 3\beta . \lambda,$$

A being a constant number, which must evidently be positive, then y, δ, ε are pure numbers, independent of the choice of units. The equations then become

$$(7) \qquad \begin{aligned} \dot{y}^2 &= \frac{1}{3}\lambda . \frac{y^4 - 3y^2 + \text{A} y + y\delta + \varepsilon}{y^2} \\ y\dot{\delta} + \dot{\varepsilon} &= 0. \end{aligned}$$

It will be shown in article 5 that there is observational

[1]) A full discussion is also contained in a paper by Professor EDDINGTON on the instability of Einstein's spherical world, which is to appear in the May number of the *Monthly Notices of the R. A. S.*

evidence that δ and ε are both very small, and A is supposed to be so chosen that also δ itself is small. LEMAÎTRE takes A $= 2$ for reasons which will appear further on.

The constant λ is an absolute constant of the dimension L^{-2}, and can be made equal to unity by an appropriate choice of the unit of length. If this is done, and if δ and ε are neglected, and A $= 2$ is adopted, the equation for y becomes

$$(7') \qquad \dot{y}^2 = \frac{y^3 - 3y + 2}{3y}.$$

2. *Integration.*

Although (7) is probably the simplest form to which the equations can be reduced, we will use a slightly more general form, which is derived from the second equation of (3) by putting

$$z = \frac{R}{R_0},$$

R_0 being a certain initial value. The equation then becomes of the form

$$(8) \qquad \dot{z}^2 = \frac{1}{3}\lambda \frac{Z^2}{z^2},$$

where Z^2 is of the fourth degree in z, the term in z^3 being absent. By an appropriate choice of R_0 we can give the four roots of the equation $Z^2 = 0$ the values $1 \pm \sqrt{-a}$ and $-1 \pm \sqrt{1-b}$. We then have

$$(9) \qquad Z^2 = (z^2 - 2z + 1 + a)(z^2 + 2z + b)$$

and

$$R_0^2 \lambda = \frac{3}{3-a-b}$$

$$(10) \qquad \frac{\alpha}{R_0} = \frac{6(1+a-b)}{3-a-b}$$

$$\frac{\beta}{R_0^2} = \frac{(1+a)b}{3-a-b}.$$

We will assume provisionally that α and β, and consequently also a and b, are constants.

Comparing with (6) we have

$$A + \delta = \frac{2(1+a-b)}{\left(1 - \frac{a+b}{3}\right)^{3/2}}, \qquad \varepsilon = \frac{(1+a)b}{\left(1 - \frac{a+b}{3}\right)^2},$$

thus A $= 2$, $\delta = \varepsilon = 0$ corresponds to $a = b = 0$. Like δ and ε, a and b are pure numbers, independent of the choice of units.

If we introduce a new unit of time, by putting

$$(11) \qquad \tau = \sqrt{\frac{\lambda}{3}} \cdot ct,$$

the relation between z and τ, i. e. between the radius of the universe and the time, is given by

$$(12) \qquad \tau - \tau_0 = \int \frac{z\,dz}{Z},$$

τ_0 being a constant. If we put further

$$(13) \qquad x = z - 1, \quad X^2 = x^2 + 4x + B^2, \quad B^2 = 3 + b,$$

the integral (12) becomes

$$(12') \qquad \tau - \tau_0 = \int \frac{(x+1)\,dx}{X\sqrt{x^2+a}},$$

which is an elliptic integral of the third kind. In the case $a = 0$ it can be integrated by logarithms, thus:

$$(14) \qquad \tau - \tau_0 = lg\,(x + X + 2) + \frac{1}{B}\,lg\,\frac{x+X-B}{x+X+B}.$$

The first term becomes positively infinite for $x = \infty$, the second term becomes negatively infinite for $x = 0$, i. e. $z = 1$, $R = R_0$. The radius of the universe thus increases from R_0 at $t = -\infty$ to infinity at $t = +\infty$, both the initial and the final value being reached asymptotically. This is the solution of LEMAÎTRE, who, however, only considers the case $b = 0$, $B = \sqrt{3}$.

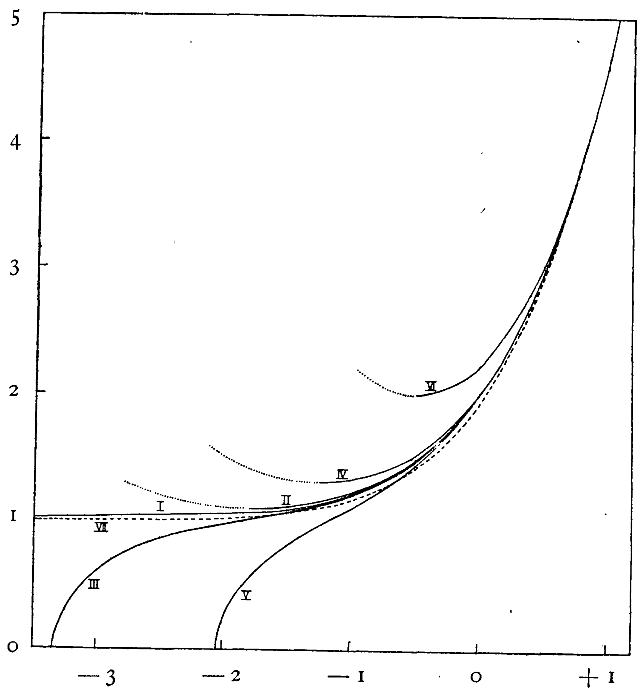
3. *Special cases.*

The condition $a = 0$ is the condition that the equation $Z^2 = 0$ shall have a double root, which, by the choice of R_0, has the value $z_0 = 1$. If a is different from zero this double root separates into two separate ones $z_0 = 1 \pm \sqrt{-a}$. If a is negative these are real, and the larger of the two, $z_0 = 1 + \sqrt{-a}$, must be taken as the lower limit of the integral: the radius increases from $R = R_0 z_0$ to $R = \infty$. If a is positive the two roots are imaginary and the lower limit of the integral must be taken zero: the radius increases from $R = 0$ to $R = \infty$. In both cases the upper limit $R = \infty$ is still reached asymptotically, but the time taken to reach any finite value R_1 from the initial value $R_0 z_0$ or 0 is finite, and of the order of magnitude of the radius R_1, or R_0, itself, i. e. of the order of 10^9 years. This short time scale would tempt us to assume, as LEMAÎTRE does, that the true value of a in the actual universe is zero. It should be remarked, however, that this does not help us very much. The logarithmic infinity of $-\tau$ does not begin to assert itself until very near the limiting value $z = 1$, and the time taken by the radius to increase from any initial value $z_0 > 1$ to a larger value z_1 is practically the same for all values of a for which the value z_0 can be reached at all. In consequence of (10) the value of a is limited to $-1 + b < a < 3 - b$.

The value of b has very little influence on the value of the integral. Evidently, by (10), b lies necessarily between the limits 0 and the smaller of the two quantities $3-a$ and $1+a$. In the actual universe it is probably very small, of the order of 10^{-5}, as will be shown below in art. 5.

In the accompanying diagram the relation between τ and z has been plotted for some special values of a and b. These are, τ_{z_0} being $\tau-\tau_0$ for $z=z_0$,

I.	$b=0$	$a=0$	$z_0=1$	$\tau_{z_0}=-\infty$
II.	$b=0$	$a=-\cdot01$	$z_0=1\cdot1$	$-3\cdot35$
III.	$b=0$	$a=+\cdot01$	$z_0=0$	$-1\cdot80$
IV.	$b=0$	$a=-\cdot1$	$z_0=1\cdot3162$	$-2\cdot05$
V.	$b=0$	$a=+\cdot1$	$z_0=0$	$-1\cdot25$
VI.	$b=0$	$a=-1$	$z_0=2$	$-0\cdot49$
VII.	$b=1$	$a=0$	$z_0=1$	$-\infty$



Relation between z and $\tau-\tau_0$.
The vertical coordinate is z, the horizontal coordinate is $\tau-\tau_0$.

The cases II, III, IV, V were treated by numerical integration. The constant of integration τ_0 was so chosen as to bring the curves into coincidence for large values of z.

The case VII, $a=0$, $b=1$ gives $\alpha=0$. This is thus a universe filled with radiation, but without any material mass. The integral (12) is in this case

$$(15)\qquad \tau-\tau_0 = lg\sqrt{(z-1)(z+1)}.$$

It will be seen that the expansion of the radius of this universe from $R=R_0$ to $R=\infty$ takes place in almost exactly the same way as in the case I, which is a universe filled with matter but without any radiation (or other pressure).

The case VI, $a=-1$, $b=0$, gives $\alpha=0$, $\beta=0$, i. e. an entirely empty universe, without either matter or radiation. The integral in this case is

$$(16)\qquad \tau-\tau_0 = lg(z+\sqrt{z^2-4}),$$

which is equivalent to

$$z = 2\cosh(\tau-\tau_0),$$

or

$$(16')\qquad R=R_1\cosh(\tau-\tau_0), \qquad R_1 = 2R_0.$$

This is identical to the solution (B), viz:

$$(17)\quad ds^2 = -R_1^2[d\rho^2+\sin^2\rho(d\psi^2+\sin^2\psi\, d\theta^2)] + R_1^2\cos^2\rho\, du^2.$$

By the transformation

$$\sin\rho = \cosh\tau.\sin\chi$$
$$\tanh u = \tanh\tau.\sec\chi$$

the line-element (17) is transformed into

$$(17')\quad ds^2 = -R_1^2\cosh^2\tau[\delta\chi^2+\sin^2\chi(d\psi^2+\sin^2\psi\, d\theta^2)] + R_1^2\, d\tau^2.$$

We have in this case, by (10), $R_0^2\lambda = \frac{3}{4}$, and consequently, by (11), $R_1 d\tau = c\,dt$. [1])

4. *Determination of the constants.*

The radius vector is $r=R\chi$. Consequently the radial velocity is

$$\frac{V}{c} = R\dot{\chi} + \chi\dot{R}.$$

If the coordinate χ has no systematic motion, which will be shown below to be the case, the systematic radial velocity is proportional to the radius vector. Thus:

$$(18)\qquad \frac{V}{cr} = \frac{\dot{R}}{R} = \frac{\dot{z}}{z}$$

We have found in *B. A. N.* 185 that the extra-galactic nebulae do show a systematic radial velocity proportional the distance, and we have determined the ratio

$$\frac{V}{cr} = \frac{1}{R_B}, \qquad R_B = 2000\ A.$$

By (8) and (18) we have thus

$$(19)\qquad \frac{1}{R_B^2} = \frac{\lambda Z_1^2}{3z_1^4},$$

the suffix 1 denoting the values at the present moment.

In EINSTEIN's solution (A) we have, on the other hand [2])

$$\frac{2}{R_A^2} = \varkappa(\rho_0+4p),$$

[1]) The solution (B) can thus be considered as a static or a non-static solution at will. That this is possible is due to the fact that in it the *four*-dimensional space is isotropic and of constant curvature.

[2]) See DE SITTER, On Einstein's theory of gravitation and its astronomical consequences, third paper, *M. N.* lxxviii (1917), p. 21, footnote. It should be noted that the quantity called ρ_0 in the formulas of that footnote is our $\rho=\rho_0+3p$.

which, with $\rho_0 = 2.10^{-28}$ gr. cm^{-3}, $p = 0$, gives

$$R_A = 2300\,A$$

and consequently, by (5) and (10),

$$(20) \qquad \frac{1}{R_A^2} = \lambda.\ \frac{3\,(1 + a - b)\,z_1 + 2\,(1 + a)\,b}{3 z_1^4}$$

Eliminating λ from (19) and (20) we find for z_1 the equation of the fourth degree

$$(21) \qquad Z_1^2 = (z_1^2 - 2z_1 + 1 + a)\ (z_1^2 + 2z_1 + b) = \frac{R_A^2}{R_B^2}\,[3\,(1 + a - b)\,z_1 + 2\,(1 + a)\,b].$$

If we put $b = 0$, the equation becomes of the third degree, viz:

$$(21') \qquad z_1^3 - (3 - a)\,z_1 + \left(2 - 3\,\frac{R_A^2}{R_B^2}\right)(1 + a) = 0,$$

and substituting the value of $R_A/R_B = 1{\cdot}15$, or very nearly

$$\frac{R_A^2}{R_B^2} = \frac{4}{3}$$

we find

$$(21'') \qquad z_1^3 - 3z_1 - 2 + a\,(z_1 - 2) = 0,$$

from which

$$(22) \qquad z_1 = 2,$$

independent of the value of a. This is an accidental circumstance. If we had taken any other value of R_A^2/R_B^2 the value of z_1 would, of course, depend on a.

Then from (19) or (20) we can find λ, and then R_0 from the first of (10). If we take $b = 0$ we have

$$(23) \qquad \lambda = \frac{1}{1 + a}\cdot\frac{z_1^3}{R_A^2}, \qquad R_0^2 = \frac{3\,(1 + a)}{3 - a}\cdot\frac{R_A^2}{z_1^3}$$

Now, if R_A^2/R_B^2 is not too small, the preponderating terms in (21′) are

$$z_1^3 \backsim 3\,\frac{R_A^2}{R_B^2}\,(1 + a)\ ,$$

or

$$\frac{R_A^2}{z_1^3} \backsim \frac{R_B^2}{3\,(1 + a)}\ ,$$

and consequently

$$(23') \qquad \lambda \backsim \frac{3}{R_B^2}\ , \quad R_0^2 \backsim \frac{R_B^2}{3 - a}\ .$$

The values of λ and R_0 thus depend almost entirely on R_B, and not on R_A, as might be suggested by (23). It has been already remarked in *B. A. N.* 185 that the uncertainty of the adopted value of R_B is probably not more than corresponds to a probable error of a fourth or a fifth of the amount, while R_A, which depends on $\sqrt{\rho_0}$, may be uncertain by a factor of 10, or more. The product $R_0^2\,\lambda$ is, of course, independent of both R_A and R_B.

Consequently R_0 and λ are known within narrow limits of uncertainty. The present radius $R_1 = R_0\,z_1$, depends on R_A, and is much less certain.

Taking $R_A^2/R_B^2 = 4/3$ we find for some of the special cases considered in the preceding article

(24)

I:	$R_0^2\lambda = 1$	, $z_1 = 2$	, $R_0 = 816A$,	$\lambda = 1{\cdot}5\ .10^{-6}A^{-2}$
II:	0·997	2	811	1·515
III:	1·003	2	819	1·485
VII:	1·5	1·623	877	1·950

For the case VI, $b = 0$, $a = -1$, we have $\rho_0 + 4p = 0$, and the formulas of the present article are not applicable. They would lead to $\lambda = \infty$, $R_0 = 0$, which is devoid of meaning.

If we take another value for ρ_0, and consequently for R_A^2/R_B^2, the values of z_1, R_0 and λ are changed, $R_0^2\,\lambda$ remaining the same. For the case I ($a = 0$, $b = 0$) taking $\rho_0 = 2.10^{-30}$, $R_A^2/R_B^2 = 400/3$, we find

$$(24') \qquad z_1 = 7{\cdot}49\,, \quad R_0 = 1126A\,, \quad \lambda = 0{\cdot}787.10^{-6}A^{-2}.$$

By (23′) the maximum value of R_0 (for very small ρ_0) is $\frac{1}{3}\,R_B\sqrt{3} = 1155A$.

As a compromise it is perhaps convenient to adopt an intermediate value, which gives $R_0 = 1000A$, $\lambda = 10^{-6}A^{-2}$ for case I. This value is

$$(25) \qquad \frac{R_A^2}{R_B^2} = 7{\cdot}2\,, \quad \rho_0 = 3{\cdot}73.10^{-29}\,.$$

The time scale depends on λ, one division in the figure on page 213 corresponding to

$$t/\tau = 1{\cdot}057\,\sqrt{(3/\lambda)}.10^6 \text{ years.}$$

The present position on the curve is given by z_1. The unit of the vertical coordinate in the figure is R_0. For the value (25) of R_A^2/R_B^2 we have, for some of the curves of the series VI-IV-II-I-III-V....., all of which have $b = 0$, while a varies from -1 to $+3$:

(26)

$a = +0{\cdot}1$ (V):	$z_1 = 3{\cdot}129$,	$R_0 = 1058A$,	$\frac{t}{\tau} = 1850$ y^r
0 (I)	3·064	1000	1830
−0·1 (III)	2·998	967	1800
−0·5	2·676	802	1585

The total material mass of the universe is

$$M = \frac{2\pi^2}{\varkappa}\,\frac{1 + a - b}{1 - \frac{1}{3}\,(a + b)}\cdot R_0\,,$$

the numerical factor being $2\pi^2/\varkappa = 1{\cdot}060.10^{28}$ gr.cm^{-1}, and is thus of the order of 10^{55} grams.

5. *Discussion of the pressure. Transformation of matter into radiation.*

The stars — or the galactic systems (nebulae) into which they are congregated — are continually radiating away energy, by which their mass is diminished. According to the formulas (15) and (16) of *B.A.N.* 185 the absolute magnitude (visual) of the spiral nebulae is $-15{\cdot}56$ and of the elliptical nebulae $-16{\cdot}06$. The latter number is too uncertain to merit any confidence. If we adopt the former, the difference in absolute magnitude with the sun is $-20{\cdot}4$, and the radiation from one spiral consequently is $1{\cdot}5 \times 10^8$ times that of the sun, or 2.10^{28} grams per year. The assumed mass was 2.10^{44} grams. Thus, if we chose the unit of time so as to make $c = 1$ corresponding to our unit of distance of $1A = 10^{24}$ cm, i. e. approximately a million years, the rate of conversion of matter into energy would be:

$$\frac{\dot{\alpha}}{\alpha} = -10^{-10}. \tag{27}$$

The radiation may have been underestimated, as we have neglected the reduction from visual to bolometric magnitude for the spirals, but, so far as the *stellar* radiation from the spirals is involved, this can hardly amount to more than one or two tenths of a magnitude, and also other radiations — penetrating radiation, etc. — cannot contribute much. The uncertainty on this account does probably not amount to more than a factor of 2. On the other hand the mass may have been overestimated. Perhaps on the whole 10^{-9} may be thought more probable than the value (27). This would correspond closely to the rate of generation of energy by a dwarf of somewhat later type than the sun.

We can measure the rate of conversion of matter into radiant energy against the rate of expansion of the universe, by putting

$$\frac{\dot{\alpha}}{\alpha} = -\gamma \frac{\dot{R}}{R}, \tag{28}$$

γ being positive. The value (27) of $\dot{\alpha}/\alpha$ gives for the value of γ at the present moment [1]) $\gamma = 2.10^{-7}$. It is not probable that γ will be a constant throughout the evolution of the universe, but as its true value for the distant past and future is unknown, the best we can do is to treat it as a constant. Consequently

$$\alpha = \alpha_0 R^{-\gamma},$$

and consequently

$$\varkappa \rho_0 = \frac{\alpha_0}{R^{3+\gamma}}. \tag{29}$$

[1]) In a paper (*Proc. Nat. Acad. of Sci.*, Washington, April 1930), which comes to hand while the present note is being prepared for the press, R. C. TOLMAN gives an approximate solution of the equations (3), which leads to the value $\gamma = 3$ (*l.c.* p. 331). This seems inadmissible. See also the last article of the present paper.

Then (4′) gives

$$\beta = \beta_0 + \frac{\alpha_0 \gamma}{3(1-\gamma)} R^{1-\gamma},$$

and consequently

$$\varkappa p = \frac{\beta_0}{R^4} + \frac{\alpha_0 \gamma}{3(1-\gamma)} \cdot \frac{1}{R^{3+\gamma}}. \tag{30}$$

The second term is the pressure of the radiation produced by the radiating matter. If we suppose that there is no radiation in the universe not emanated from matter, then the first term represents the kinematical pressure, corresponding to the random motions of the extragalactic nebulae, considered as the molecules of a gas. According to HUBBLE, and to our own finding in *B.A.N.* 185, these random motions are of the order of 150 km/sec, or $0{\cdot}5.10^{-3}$. The ratio p/ρ_0 [1]) between the kinematic pressure and the density is equal to the square of this, or $\frac{1}{4}.10^{-6}$. Consequently, if we neglect the second term of (30), we have by (10) for the present moment:

$$\frac{(1+a)\,b}{6(1+a-b)\,z_1^{1-\gamma}} = \frac{1}{4} \cdot 10^{-6}.$$

It follows that b is of the order of magnitude of 10^{-6} or 10^{-5}.

Neglecting the second term of (30), i. e. treating α and β as constants, we have

$$\frac{p}{\rho} = \frac{const.}{R}, \tag{31}$$

and consequently the random motions should decrease as $1/\sqrt{R}$.

Turning now to the second term of (30), we find that the total amount of radiation pressure in the universe, so far as it originates from the radiation of the stars and other matter, is $\pi^2 R^3 p$, or

$$\frac{\pi^2}{\varkappa} \frac{\alpha_0 \gamma}{3(1-\gamma)} R^{-\gamma},$$

and consequently, since γ is positive, the total amount of radiation *decreases* as a consequence of the conversion of matter into radiation. The explanation of this paradox is simple. By the adiabatic expansion of the universe the pressure is diminished, and this more than counterbalances the increase by the conversion of matter into radiation. This can easily be verified by making up the account of loss and gain of energy. If we denote the total amount of radiative energy in the universe by E, and the total material mass by M, the change of energy is

$$\dot{E} = \dot{E}_1 + \dot{E}_2,$$

[1]) Strictly speaking we should use the ratio $p/(\rho_0 + 4p)$ (see EDDINGTON, Mathematical theory of relativity, p. 122).

where $\dot{E}_1$ is the gain by the conversion of matter, thus

$$\dot{E}_1 = -\dot{M} = \gamma M \frac{\dot{R}}{R},$$

and $\dot{E}_2$ is the loss by degradation consequent on the increase of wavelength corresponding to the receding velocities by Doppler's principle. Thus, by Planck's equation $E = h\nu$, we have

$$\frac{\dot{E}_2}{E} = \frac{\dot{\nu}}{\nu} = -\frac{\dot{R}}{R},$$

as will be shown in the next article. Consequently

$$\dot{E} = \frac{\dot{R}}{R}(\gamma M - E),$$

or, since $M = \pi^2 R^3 \rho_0/\varkappa$, by (29)

$$E\dot{R} + R\dot{E} = \gamma M\dot{R} = \frac{\pi^2}{\varkappa}\alpha_0 \gamma R^{-\gamma} \dot{R},$$

from which

$$E = \frac{\pi^2}{\varkappa}\frac{\alpha_0 \gamma}{1-\gamma} R^{-\gamma}, \tag{32}$$

which, by $\pi^2 R^3 p/\varkappa = \frac{1}{3} E$, is identical with the second term of (30).

The new theory thus incidentally gives an answer to the old question what becomes of the energy that is continually being poured out into space by the stars. It is used up, and more than used up, by the work done in expanding the universe. Nevertheless it would not be correct to say that the universe is expanded *by* the radiation pressure. It would expand just the same if γ were zero, i.e. if no radiation was emitted by matter. The expansion is due to the constant λ.

In the integration of the differential equation for ER we have omitted to add a constant of integration. This would in (32) give an additional term

$$\frac{E_0}{R},$$

which represents the initial energy of radiation, if any, not emanated from matter, and forms part of the first term of (30). This, like the kinematical pressure, diminishes proportional to $1/R$ by the adiabatic expansion of the universe.

It should be kept in mind that the approximation $\gamma = $ constant is not supported (nor contradicted) by any observational evidence. The formulae derived in the present article can thus only be considered as describing the state and rate of change of the universe at the present moment, and must not be used for extrapolation into the distant past or future.

6. *Rays of light.*

We have seen in the preceding article that $\dot{\alpha}$ is very small. We can thus, in discussing the phenomena in the actual universe at the present moment, as a good approximation treat α and β, and consequently also a and b, as constants, as was already done above. Also we can, with sufficient approximation, take $b = 0$.

For a ray of light $ds = 0$, and therefore $d\sigma = c\,dt/R$. The equation of a ray travelling between two points characterised by the coordinates σ and σ_1 is thus

$$\sigma_1 - \sigma = c\int_t^{t_1} \frac{dt}{R} = \sqrt{\frac{3}{R_0^2 \lambda}} \int_z^{z_1} \frac{dz}{Z}, \tag{33}$$

where z_1 and z are the radii of the universe, in R_0 as unit, at the times when the light travelled through the points σ_1 and σ respectively. In the case I of art. 3 ($a = 0$, $b = 0$) we have thus

$$\sigma_1 - \sigma = \frac{1}{\sqrt{R_0^2 \lambda}} \left| lg \frac{x + X - \sqrt{3}}{x + X + \sqrt{3}} \right|_z^{z_1}. \tag{34}$$

Consider two successive light pulses leaving σ at a time interval δt. From (33) we have

$$\frac{\delta t_1}{R_1} - \frac{\delta t}{R} = 0,$$

or

$$\frac{\delta t_1}{\delta t} = \frac{\nu}{\nu_1} = \frac{\lambda_1}{\lambda} = \frac{R_1}{R},$$

where ν and λ are the emitted, ν_1 and λ_1 the observed frequency and wavelength, if the source and the observer are at σ and σ_1 respectively. Therefore

$$\frac{\lambda_1 - \lambda}{\lambda} = \frac{R_1 - R}{R} = \frac{z_1 - z}{z}, \tag{35}$$

from which we can find the radius z, if z_1 is given. The emitted wavelength λ, of course, is a constant for every particular kind of radiation. The formula (35) thus gives the displacement towards the red ($\lambda_1 > \lambda$) for light travelling from the past towards the future ($z_1 > z$). For relatively short intervals we can take

$$R_1 - R = \delta R = \dot{R}.c\delta t = \dot{R}.r,$$

r being the distance travelled. Further we then have, by Doppler's principle,

$$\frac{V}{c} = \frac{\delta\lambda}{\lambda},$$

and consequently

$$\frac{V}{cr} = \frac{\dot{R}}{R}. \tag{18}$$

For larger intervals the distance computed by (18) will be much smaller than the correct value found from (33) with the value of z derived from (35). For the case I ($a = 0$) of art. **3** we have the following comparison between $\sigma' = r/R$ computed by (18) and $-\sigma$ by (35) and (34). For z_1 we have taken $z_1 = 3$ and to convert τ into years we have used the corresponding factor 1850 [1]), τ itself being computed by (14).

z	$\frac{V}{c} = \frac{\delta\lambda}{\lambda}$	σ'	$\mp\sigma$	τ	t 10^9 years
1	2	1·342	∞	− ∞	− ∞
1·05	1·857	1·246	3·900	− 2·971	− 5·49
1·1	1·636	1·098	2·232	− 1·980	− 3·66
1·5	1	0·671	0·843	− ·983	− 1·82
2	0·5	0·335	0·369	− ·514	− 0·95
2·5	0·2	0·134	0·140	− ·220	− 0·41
3	0	0	0	0	0
3·5	(− 0·143)	(0·096)	0·090	+ ·167	+ 0·31
∞	(− 1)	(0·671)	0·607	+ ∞	+ ∞

When the distance $-\sigma$ given in the fourth column exceeds $\pi = 3·141$, the light has travelled entirely round the world before reaching us. It will be seen that light emitted after the epoch corresponding to the radius z, which makes $\sigma_1 - \sigma = \pi$ in (33) with $z_1 = \infty$, can never complete the circuit of the world, the expansion being too quick for light to overtake it by a complete circuit.

For the case I this value is $z = 1·0728$. For the case II light which is emitted at the origin of time ($z = 1·1$) can never travel a further distance than $0·68\,\pi$ ($z_1 = \infty$), which is reached asymptotically for $t = \infty$.

It is important to note that the velocity of light is independent of the space-coordinates. In the coordinates χ, ψ, θ the rays of light are geodesics described with constant velocity. It follows that on triangles formed by rays of light the ordinary spherical trigonometry is applicable. Consequently the parallax is given by

$$p = \frac{a}{\tan\chi},$$

and the apparent diameter by

$$\delta = \frac{d}{\sin\chi}.$$

The intensity of light is given by

$$I = \frac{I_0}{\sin^2\chi}.$$

Consequently the diameter and the magnitude still follow *the same* law, log sin χ taking the place of log r in the formulas of *B. A. N.* 185, as well for log r_d as log r_m.

[1]) This corresponds to $\rho_0 = 4·10^{-29}$, $R_A{}^2/R_B{}^2 = 20/3$, instead of (25).

7. *Motion of a material particle.*

The equations of motion of a material point under the influence of inertia alone in the expanding world can easily be derived. The Christoffel symbols for the coordinates χ, ψ, θ, ct are:

$$\begin{Bmatrix}22\\1\end{Bmatrix} = -\sin\chi\cos\chi \qquad \begin{Bmatrix}33\\1\end{Bmatrix} = -\sin\chi\cos\chi\sin^2\psi$$

$$\begin{Bmatrix}12\\2\end{Bmatrix} = \begin{Bmatrix}13\\3\end{Bmatrix} = \cos\chi \qquad \begin{Bmatrix}33\\2\end{Bmatrix} = -\sin\psi\cos\psi \qquad \begin{Bmatrix}23\\3\end{Bmatrix} = \cot\psi$$

$$\begin{Bmatrix}14\\1\end{Bmatrix} = \begin{Bmatrix}24\\2\end{Bmatrix} = \begin{Bmatrix}34\\3\end{Bmatrix} = \frac{\dot R}{R}$$

$$\begin{Bmatrix}11\\4\end{Bmatrix} = R\dot R \qquad \begin{Bmatrix}22\\4\end{Bmatrix} = R\dot R\sin^2\chi \qquad \begin{Bmatrix}33\\4\end{Bmatrix} = R\dot R\sin^2\chi\sin^2\psi$$

We can, as always, suppose $\psi = \frac{1}{2}\pi$, $d\psi/ds = 0$, and consider the coordinates χ, θ only. The equations for the geodesic then become

$$\frac{d^2\chi}{ds^2} - \sin\chi\cos\chi\left(\frac{d\theta}{ds}\right)^2 = -2\frac{\dot R}{R}\frac{d\chi}{ds}\frac{d(ct)}{ds}$$

$$\frac{d^2\theta}{ds^2} + 2\cot\chi\frac{d\chi}{ds}\frac{d\theta}{ds} = -2\frac{\dot R}{R}\frac{d\theta}{ds}\frac{d(ct)}{ds}.$$

Putting

$$\varphi^2 = \left(\frac{d\sigma}{ds}\right)^2 = \left(\frac{d\chi}{ds}\right)^2 + \sin^2\chi\left(\frac{d\theta}{ds}\right)^2$$

$$\Gamma = \sin^2\chi\frac{d\theta}{ds},$$

we derive the two equations

$$\frac{d(\varphi^2)}{ds} = -4\frac{\dot R}{R}\varphi^2\frac{d(ct)}{ds}$$

$$\frac{d\Gamma}{ds} = -2\frac{\dot R}{R}\Gamma\frac{d(ct)}{ds},$$

from which we find easily

$$\varphi = \frac{\varphi_0}{R^2}, \qquad \Gamma = \frac{\Gamma_0}{R^2},$$

corresponding to the integrals of energy and areas in the stationary universe.

Eliminating ds, we have for the differential equation of the track:

$$\frac{\varphi^2}{\Gamma^2} = \frac{d\chi^2 + \sin^2\chi\,d\theta^2}{\sin^4\chi\,d\theta^2} = \text{const.} = \text{cosec}^2\chi_0,$$

giving

(36) $$\tan\chi\cos(\theta - \theta_0) = \tan\chi_0,$$

which is the equation of a geodesic (great circle) in three-dimensional space. To find the velocity, we must introduce the time instead of the interval ds. We have

$$\left(\frac{d(ct)}{ds}\right)^2 = 1 + R^2\left(\frac{d\sigma}{ds}\right)^2 = 1 + R^2\varphi^2.$$

Thus, putting

$$v^2 = \left(\frac{d\chi}{d\,(ct)}\right)^2 + \sin^2\chi\left(\frac{d\theta}{d\,(ct)}\right)^2 = \left(\frac{d\sigma}{d\,(ct)}\right)^2$$

$$G = \sin^2\chi\,\frac{d\theta}{d\,(ct)} \quad ,$$

we have

$$\varphi = v\,\frac{d\,(ct)}{ds}\,, \qquad \Gamma = G.\frac{d\,(ct)}{ds}\,.$$

The velocity v is thus given by

$$(37) \qquad (R\,v)^2 = \frac{\varphi_0^2}{R^2 + \varphi_0^2}.$$

The random velocities are thus decreasing, and unless φ_0 is very large, are practically proportional to $1/R$. This is in contradiction with the result found in art. 5 (formula (31), page 215) that the random velocities decrease as $1/\sqrt{R}$. The formula (31), of course, supposes α and β to be constant, of which assumption the present result is independent. But this can hardly be the complete explanation of the paradox. The question must be left open for the present.

For a ray of light we have, of course, $\varphi_0 = \infty$, and (37) gives $v = 1/R$, as has been found already (art. 6).

It should be noted that the results of this article are entirely independent of the integration of the differential equation for R, and of the assumption that α and β are constant.

8. *Miscellaneous remarks.*

In an interesting paper,[1]) which was published while the present communication was being prepared for the press, Professor R. C. TOLMAN independently derives the same equations that are used by LEMAÎTRE. If in his equations (34) $R.e^{\frac{1}{2}g}$ is called R, they become identical with our equations (3). Using the notation of the present paper, TOLMAN's solution is $R = R_0\,e^{kt}$. If to (3) we add the condition $k =$ constant, or

$$\frac{d^2}{dt^2}(lg\,R) = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = 0$$

they can be completely integrated. We find easily (with the help of (4′), if desirable)

$$(38) \qquad \begin{aligned} \alpha &= 6\,R - 4\,\beta_0\,R^3 \\ \beta &= \beta_0\;R^4 - R^2 \\ R &= R_1\,e^{k\,(t-t_1)} \\ 3\,k^2 &+ \beta_0 = \lambda. \end{aligned}$$

If we add the further condition, as TOLMAN does, that $\beta = 0$ for $t = t_1$, $R = R_1$, we have $\beta_0 = 1/R_1^2$, and, for the same $t = t_1$,

$$\left(\frac{\dot{\alpha}}{\alpha}\right)_1 = -3\left(\frac{\dot{R}}{R}\right)_1.$$

The set of formulas (38) does not look very attractive. The point is that $\ddot{R}/R$ naturally is of the same order of magnitude as $(\dot{R}/R)^2$, and, since only $(\dot{R}/R)^2$ and not $\dot{R}/R$ itself occurs in the equations, we can not expect to get a good approximation by simply putting $\ddot{R}/R = \dot{R}^2/R^2$.

The important point is that we must have a ***non-static*** solution of the differential equations of the inertial field. This leads to an expanding world, the radius of curvature being a function of the time, e. g. one of the curves of the diagram of page 213. There is nothing in our observational data to determine the choice of any particular curve for the representation of the history of the universe. The selection must remain a matter of taste, or of philosophical preference. The case I gives a logarithmically infinite time elapsed since the beginning of the expansion. In any of the other cases, however, the same end can easily be obtained by introducing another time variable, such as

$$t' = k\;lg\,\frac{R - R_0}{R_1 - R_0}\,, \quad k = \sqrt{\frac{3}{\lambda}}\cdot\frac{z_1\,(z_1 - z_0)}{Z_1}\,.$$

the factor k being determined by the condition $dt'/dt = 1$ for $R = R_1$.

Also, if α and β are not treated as constants, the integral (12), taken between the limits z_0 and z, can easily be made infinite, by so choosing one of the parameters, which must occur in the expression of α (or β) as a function of z, that z_0 is a double root of the equation $Z_1^2 = 0$. Here also, however, the infinity would be only logarithmic.

If a is negative, i. e. if we have one of the curves like II or IV in the diagram, we can, of course, also suppose that the radius, being infinite for $t = -\infty$, has begun to shrink, passing though a minimum value R_0 about at the time of the birth of our planetary system, and is now increasing to become infinite again at $t = +\infty$. Although this would have some advantages from the point of view of the collision theory of the origin of the planetary system, by making the distances small and the velocities large [by (37)] at the time when the collision is required, it does not, on the whole, look very probable.

It should also be kept in mind that many simplifying assumptions have been introduced, which, though perfectly legitimate for the present moment, may make extrapolation to the distant past or future unsafe.

[1]) The effect of annihilation of matter on the wave-length of light from the nebulae, *Proceedings Nat. Acad. Sci. Wash.* **16**, 320 (April 1930).