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# Universality of Brézin and Zee's spectral correlator

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## Abstract

The smoothed correlation function for the eigenvalues of large hermitian matrices, derived recently by Brézin and Zee [Nucl Phys B402 (1993) 613], is generalized to all random-matrix ensembles of Wigner–Dyson type

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## 1. Introduction

A basic problem in random-matrix theory is to compute the correlation of the eigenvalue density at two points in the spectrum, from the Wigner–Dyson probability distribution of the eigenvalues [1]. The correlation is a manifestation of the level repulsion resulting from the jacobian  $\prod_{i < j} |\lambda_i - \lambda_j|^\beta$ , which is associated with the transformation from the space of  $N \times N$  hermitian matrices to the smaller space of  $N$  eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_N$ . [The power  $\beta$  depends on whether the matrix elements are real ( $\beta = 1$ , orthogonal ensemble), complex ( $\beta = 2$ , unitary ensemble), or quaternion real ( $\beta = 4$ , symplectic ensemble)] The Wigner–Dyson probability distribution

$$P(\{\lambda_n\}) = Z^{-1} \exp[-\beta W(\{\lambda_n\})], \quad (1)$$

with  $Z$  a normalization constant and

$$W(\{\lambda_n\}) = - \sum_{i < j} \ln |\lambda_i - \lambda_j| + \sum_i V(\lambda_i), \quad (2)$$

describes an ensemble where *all* eigenvalue correlations are due to the jacobian. The potential  $V(\lambda)$  determines the mean density  $\rho(\lambda)$  of the eigenvalues, which is non-zero in some interval  $(a, b)$ .

In many applications of random matrix theory, it is sufficient to know the eigenvalue correlations in the bulk of the spectrum, far from the end points at  $a$  and  $b$ . In some

applications, however, the presence of an edge in the spectrum is an essential part of the problem, and its effect on the spectral correlations cannot be ignored. The application to "universal conductance fluctuations" in mesoscopic conductors is one example [2-5]. The application to random surfaces and two-dimensional quantum gravity is another example [6-8].

Recently, Brézin and Zee [9,10] reported a remarkably simple result for the two-level cluster function

$$T_2(\lambda, \mu) = - \left\langle \sum_{i \neq j} \delta(\lambda - \lambda_i) \delta(\mu - \lambda_j) \right\rangle + \rho(\lambda) \rho(\mu), \quad (3)$$

which included the effects of an upper and lower bound on the spectrum. (Here  $\langle \dots \rangle$  denotes an average with distribution (1), and  $\rho(\lambda) = \langle \sum_i \delta(\lambda - \lambda_i) \rangle$  is the mean eigenvalue density.) For  $N \gg 1$ , the correlation function (3) oscillates rapidly on the scale of the spectral band width ( $a, b$ ). These oscillations are irrelevant when integrating over the spectrum, so that in the large- $N$  limit it is sufficient to know the smoothed correlation function. Brézin and Zee considered the unitary ensemble ( $\beta = 2$ ), with  $V(\lambda) = \sum_{k=1}^p c_k \lambda^{2k}$  an even polynomial function of  $\lambda$ , so that  $a = -b$ . (The case  $a \neq b$  can then be obtained by translation of the entire spectrum.) Using the method of orthogonal polynomials [1], they rigorously proved the following result for the smoothed correlation function:

$$T_2(\lambda, \mu) = \frac{1}{2\pi^2} \frac{1}{(\lambda - \mu)^2} \frac{a^2 - \lambda\mu}{[(a^2 - \lambda^2)(a^2 - \mu^2)]^{1/2}}. \quad (4)$$

The purpose of the present paper is to show how Eq. (4) can be generalized to arbitrary (non-polynomial, non-even) potentials  $V(\lambda)$ , and to all three symmetry classes ( $\beta = 1, 2, 4$ ). This universality is achieved by a functional derivative method [3], which provides a powerful (albeit non-rigorous) alternative to the classical method of orthogonal polynomials. In Ref. [3] we applied this method to the case  $a = 0, b \rightarrow \infty$  of a *single* spectral boundary. Here we extend the analysis to include a finite upper *and* lower bound on the spectrum.

## 2. Method of functional derivatives

We consider the two-point correlation function

$$K_2(\lambda, \mu) = - \left\langle \sum_{i, j} \delta(\lambda - \lambda_i) \delta(\mu - \lambda_j) \right\rangle + \rho(\lambda) \rho(\mu) \quad (5)$$

(note the unrestricted sum over  $i$  and  $j$ ), which is related to the two-level cluster function (3) by

$$K_2(\lambda, \mu) = T_2(\lambda, \mu) - \rho(\lambda) \delta(\lambda - \mu). \quad (6)$$

For  $\lambda \neq \mu$  the two correlation functions coincide, so that we can compare with Eq. (4). We prefer to work with  $K_2$  instead of  $T_2$  for a technical reason: Smoothing, in combination with the large- $N$  limit, introduces a spurious non-integrable singularity in  $T_2(\lambda, \mu)$  at  $\lambda = \mu$ , while  $K_2(\lambda, \mu)$  remains integrable<sup>1</sup>.

Our analysis is based on the exact relation [3] between the two-point correlation function  $K_2(\lambda, \mu)$  and the functional derivative of the eigenvalue density  $\rho(\lambda)$  with respect to the potential  $V(\mu)$ ,

$$K_2(\lambda, \mu) = \frac{1}{\beta} \frac{\delta \rho(\lambda)}{\delta V(\mu)}. \quad (7)$$

The smoothed correlator is obtained by evaluating the functional derivative using the asymptotic ( $N \rightarrow \infty$ ) integral relation between  $V$  and  $\rho$ ,

$$\mathcal{P} \int_a^b d\mu \frac{\rho(\mu)}{\lambda - \mu} = \frac{d}{d\lambda} V(\lambda), \quad a < \lambda < b. \quad (8)$$

(The symbol  $\mathcal{P}$  denotes the principal value of the integral.) Corrections to Eq. (8) are smaller by an order  $N^{-1}$  for  $\beta = 1$  or 4, and by an order  $N^{-2}$  for  $\beta = 2$  [11]. Variation of Eq. (8) gives

$$\delta b \frac{\rho(b)}{\lambda - b} - \delta a \frac{\rho(a)}{\lambda - a} + \mathcal{P} \int_a^b d\mu \frac{\delta \rho(\mu)}{\lambda - \mu} = \frac{d}{d\lambda} \delta V(\lambda), \quad (9)$$

with the constraint

$$\int_a^b d\lambda \delta \rho(\lambda) = 0 \quad (10)$$

(since the variation of  $\rho$  is to be carried out at constant  $N$ ). The end point  $a$  is either a fixed boundary, in which case  $\delta a = 0$ , or a free boundary, in which case  $\rho(a) = 0$ . Similarly, either  $\delta b = 0$  or  $\rho(b) = 0$ . We conclude that we may disregard the first two terms in Eq. (9), containing the variation of the end points. What remains is the singular integral equation

$$\mathcal{P} \int_a^b d\mu \frac{\delta \rho(\mu)}{\lambda - \mu} = \frac{d}{d\lambda} \delta V(\lambda), \quad a < \lambda < b, \quad (11)$$

which we need to invert in order to obtain the functional derivative  $\delta \rho / \delta V$ .

The general solution to Eq. (11) is [12]

<sup>1</sup> The distinction between  $T_2(\lambda, \mu)$  and  $K_2(\lambda, \mu)$  was not made explicitly in Ref. [9], because only the case  $\lambda \neq \mu$  was considered.

$$\delta\rho(\lambda) = \frac{1}{\pi^2} \frac{1}{[(\lambda - a)(b - \lambda)]^{1/2}} \times \left( C - \mathcal{P} \int_a^b d\mu \frac{[(\mu - a)(b - \mu)]^{1/2}}{\lambda - \mu} \frac{d}{d\mu} \delta V(\mu) \right). \tag{12}$$

The coefficient  $C$  is determined by

$$C = \pi \int_a^b d\lambda \delta\rho(\lambda). \tag{13}$$

In view of Eq. (10), we have  $C = 0$ . Combination of Eqs. (7) and (12) yields the two-point correlation function

$$K_2(\lambda, \mu) = \frac{1}{\beta\pi^2} \frac{1}{[(\lambda - a)(b - \lambda)]^{1/2}} \times \frac{\partial}{\partial\lambda} \frac{\partial}{\partial\mu} \left( [(\mu - a)(b - \mu)]^{1/2} \ln|\lambda - \mu| \right). \tag{14}$$

(We have substituted the representation  $\mathcal{P}x^{-1} = (d/dx) \ln|x|$  for the principal value.)

The two-point correlation function (14) has an *integrable* singularity for  $\lambda = \mu$ . For  $\lambda \neq \mu$  one can carry out the differentiations, with the result

$$K_2(\lambda, \mu) = \frac{1}{\beta\pi^2} \frac{1}{(\lambda - \mu)^2} \times \frac{\frac{1}{2}(a + b)(\lambda + \mu) - ab - \lambda\mu}{[(\lambda - a)(b - \lambda)(\mu - a)(b - \mu)]^{1/2}} \quad \text{if } \lambda \neq \mu. \tag{15}$$

For  $a = -b$  and  $\beta = 2$  we recover the formula (4) of Ref. [9] for even polynomial potentials in the unitary ensemble. For  $a = 0$  and  $b \rightarrow \infty$  we recover the correlator of Ref. [3],

$$K_2(\lambda, \mu) = \frac{1}{\beta\pi^2} \frac{\partial}{\partial\lambda} \frac{\partial}{\partial\mu} \ln \left| \frac{\sqrt{\lambda - \sqrt{\mu}}}{\sqrt{\lambda + \sqrt{\mu}}} \right| = \frac{1}{2\beta\pi^2} (\lambda - \mu)^{-2} (\lambda + \mu) (\lambda\mu)^{-1/2} \quad \text{if } \lambda \neq \mu, \tag{16}$$

for the case of a single spectral edge.

An important application of the smoothed two-point correlation function is to compute the large- $N$  limit of the variance  $\text{Var } A \equiv \langle A^2 \rangle - \langle A \rangle^2$  of a linear statistic  $A = \sum_{n=1}^N a(\lambda_n)$  on the eigenvalues, by means of the relationship

$$\text{Var } A = - \int_a^b d\lambda \int_a^b d\mu a(\lambda) a(\mu) K_2(\lambda, \mu). \tag{17}$$

Substituting Eq. (14) we obtain, upon partial integration,

$$\text{Var } A = \frac{1}{\beta\pi^2} \mathcal{P} \int_a^b d\lambda \int_a^b d\mu \left[ \frac{(\mu - a)(b - \mu)}{(\lambda - a)(b - \lambda)} \right]^{1/2} \frac{a(\lambda)}{\lambda - \mu} \frac{d}{d\mu} a(\mu). \quad (18)$$

Note that here it is essential to work with the expression (14) for  $K_2(\lambda, \mu)$ , which is integrable, and that one cannot use the expression (15), which has a spurious non-integrable singularity at  $\lambda = \mu$ . The formula (18) is the generalization to a spectrum bounded from above and below of previous formulas by Dyson and Mehta [13] (for an unbounded spectrum) and by the author [3] (for a spectrum bounded from below).

### 3. Conclusion

The result (14) for the smoothed two-point correlation function in the large- $N$  limit holds for *all* random-matrix ensembles of Wigner–Dyson type, i.e. with a probability distribution of the general form (1). The form of the eigenvalue potential  $V(\lambda)$  is irrelevant. It is also irrelevant whether the end point at  $\lambda = a$  (or at  $b$ ) is a fixed or a free boundary. This is remarkable, because the eigenvalue density behaves entirely different in the two cases: At a fixed boundary  $\rho(\lambda)$  *diverges* as  $(\lambda - a)^{-1/2}$ , while at a free boundary  $\rho(\lambda)$  *vanishes* as  $(\lambda - a)^{1/2}$ . Both cases are of interest for applications: The spectrum considered in Ref. [9], in connection with two-dimensional gravity, has free boundaries; The spectrum considered in Ref. [3], in connection with mesoscopic conductors, has a fixed boundary.

While the form of the eigenvalue potential is irrelevant, the form of the eigenvalue interaction does matter. Consider an eigenvalue distribution function of the form (1), but with a *non-logarithmic* eigenvalue interaction  $u(\lambda, \mu) \neq \ln|\lambda - \mu|$ . Such a distribution describes the energy level statistics of disordered metal particles [14], and the statistics of transmission eigenvalues in disordered metal wires [15]. The analysis of Sect. 2 carries over to this case, but the integral kernel  $(\lambda - \mu)^{-1}$  in Eqs. (8) and (11) has to be replaced by the kernel  $\partial u/\partial \lambda$ . The two-point correlation function now equals  $1/\beta$  times the inverse of this integral kernel, and differs from the result (14) for a logarithmic interaction.

So far we have only considered the two-point correlation function  $K_2 = \beta^{-1} \delta\rho/\delta V$  and the closely related two-level cluster function  $T_2$ . Brézin and Zee [9] also computed the three- and four-level cluster functions, and found that they vanish identically upon smoothing. The linearity of the relation (8) between  $\rho$  and  $V$  implies in fact that this holds for *all* higher-order cluster functions. This argument is equivalent to Politzer's proof [16] that any linear statistic on the eigenvalues has a gaussian distribution in the large- $N$  limit.

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