

Momentum relaxation in holographic massive gravity

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(Received 29 August 2013; published 17 October 2013)

We study the effects of momentum relaxation on observables in a recently proposed holographic model in which the conservation of momentum in the field theory is broken by the presence of a bulk graviton mass. In the hydrodynamic limit, we show that these effects can be incorporated by a simple modification of the energy-momentum conservation equation to account for the dissipation of momentum over a single characteristic time scale. We compute this time scale as a function of the graviton mass terms and identify the previously known “wall of stability” as the point at which this relaxation time scale becomes negative. We also calculate analytically the zero temperature AC conductivity at low frequencies. In the limit of a small graviton mass this reduces to the simple Drude form, and we compute the corrections to this which are important for larger masses. Finally, we undertake a preliminary investigation of the stability of the zero temperature black brane solution of this model, and rule out spatially modulated instabilities of a certain kind.

DOI: [10.1103/PhysRevD.88.086003](https://doi.org/10.1103/PhysRevD.88.086003)

PACS numbers: 11.25.Tq, 04.50.Kd, 72.10.-d

I. INTRODUCTION

Real materials, which are composed of electrons, atoms and so on, do not possess spatial translational invariance, and so momentum in these systems is not conserved. While the breaking of this symmetry—for example by the presence of a lattice or of impurities—is unimportant for some observables, it can give rise to important, qualitative effects in others. One simple example of such an observable is the conductivity in a system with a nonzero density of a conserved charge. Under an applied field, the charge carriers of this system will accelerate indefinitely if there is no way for them to dissipate momentum, leading to an infinite DC conductivity. If, however, spatial translational invariance is broken and therefore momentum dissipation is possible, the DC conductivity is finite and the delta function peak at the origin of the AC conductivity spreads out.

If holographic theories are to have a chance of explaining the experimentally determined behavior of these kinds of observables, then it is important that they can incorporate the dissipation of momentum. There have been numerous studies of the transport properties of holographic theories which dissipate momentum by a variety of methods, such as by explicitly breaking the translational symmetry of the field theory state [1–13], by including a parametrically large amount of neutral matter [14–16], or by coupling to impurities [17,18]. The recent work [19] addresses some of the general effects of momentum relaxation without reference to a specific underlying mechanism causing it, and in [20] it was recently shown how one obtains a finite DC conductivity for current flows of a certain kind in translationally-invariant two-layer systems.

A holographic model was proposed recently in which, by giving a mass to the graviton, momentum is no longer conserved [21]. This mass breaks the diffeomorphism

invariance of the gravitational theory and therefore, via the holographic dictionary, it violates the conservation of energy-momentum in the dual field theory. One reason that this approach is attractive is that it is relatively simple from a practical point of view—for example, the relevant black brane solutions are known analytically. This simplicity makes it an excellent toy model for studying the properties of holographic states of matter without momentum conservation. Theories of massive gravity are liable to be inconsistent (see, for example, [22–27] and references therein) and we do not know the microscopic details of the dual field theory (if one exists). As emphasised in [21], one could heuristically view this model as an effective theory arising from coarse-graining over the microscopic details of a bulk lattice or of impurities such that at long distances their only effect is to introduce effective mass terms for the graviton. We will address this comparison further when we present our results.

In this paper, we investigate in detail the effects of the nonconservation of momentum upon observable properties of the field theory state dual to the massive gravity solution of [21]. First, we outline how the extra bulk degrees of freedom produced by including a nonzero graviton mass result in the violation of the usual field theory Ward identities arising from momentum conservation. We then study this field theory in the hydrodynamic limit—that is, when the temperature T is much larger than the frequency ω and momentum k of any excitation, and much larger than the rate at which momentum is dissipated. We propose that in this fluidlike limit, the low energy dynamics of the theory are governed by a modified conservation law for energy-momentum T^{ab} such that for small perturbations around the equilibrium state where the fluid is at rest

$$\partial_a T^{at} = 0, \quad \partial_a T^{ai} = -(\epsilon + P)\tau_{\text{rel}}^{-1}u^i = -\tau_{\text{rel}}^{-1}T^{ti}, \quad (1)$$

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where ϵ , P and u^i are the energy density, pressure and velocity of the near-equilibrium field theory state, and the constant τ_{rel} is the characteristic time scale of momentum relaxation in the theory. We verify this proposal by comparing the low-energy transverse excitations of this modified hydrodynamics to those computed from the massive gravity theory, and determine that the hydrodynamic momentum relaxation time scale is given by

$$\tau_{\text{rel}}^{-1} = \frac{s}{2\pi(\epsilon + P)}(m_\alpha^2 + m_\beta^2), \quad (2)$$

where s is the entropy density of the field theory and m_α^2 and m_β^2 (which, *a priori*, may be negative) are the two independent mass terms for the graviton. The condition $\tau_{\text{rel}} = 0$ is equivalent to the ‘‘wall of stability’’ found in [21]. We can therefore give a physical meaning to this instability: the state is unstable when $\tau_{\text{rel}} < 0$ because it *absorbs* momentum at a constant rate, rather than dissipating it, and thus small perturbations of the state will grow exponentially in time.

When the chemical potential μ of the theory is nonzero, the relaxation of momentum affects the dynamics of the conserved U(1) current J^a , and this is the second topic that we study. We compute analytically the low frequency AC conductivity $\sigma(\omega)$ of the state when $T = 0$, $\mu \neq 0$ and $m_\alpha = 0$ and find that

$$\sigma(\omega) = \frac{\sigma_{\text{DC}} + \dots}{1 + \kappa_1 i\omega \log\left(\frac{\omega r_0}{6 - m_\beta^2 r_0^2}\right) + \kappa_2 \omega + \kappa_3 i\omega + \dots}, \quad (3)$$

where σ_{DC} is the m_β -dependent DC conductivity given in equation (66), κ_i are ω -independent and m_β -dependent quantities given in Eq. (62) and where the ellipses denote higher order terms in ω . In the limit of small graviton mass $m_\beta^2/\mu^2 \ll \omega/\mu \ll 1$, where momentum conservation is violated in a minor way, the conductivity is equivalent to that predicted by the simple Drude model

$$\sigma(\omega) = \frac{\sigma_{\text{DC}}}{1 - i\omega\tau_{\text{rel}}}, \quad (4)$$

where τ_{rel} is in fact given by the naive $T = 0$ limit of the hydrodynamic formula (2). However, the full expression (3) includes corrections to the Drude model. The inclusion of these corrections results in a transfer of spectral weight from the Drude peak to higher frequencies, and a reduction in the phase of σ from the Drude value. Over a range of intermediate frequencies it was shown numerically in [21] that the conductivity exhibits an approximate scaling law which, with the appropriate choice of m_β^2 , is similar to that seen in holographic lattice models [1,2] and is also reminiscent of that measured in the normal phase of some high- T_c superconductors [28]. Unfortunately, the expression (3) for the conductivity is perturbative in ω and contains no hints of this scaling behavior seen at (relatively) high ω .

Finally, we make a preliminary investigation of the possible existence of instabilities of the $T = 0$ state to a spatially modulated phase by computing the k -dependent masses of the bulk field excitations transverse to the momentum flow in the near-horizon AdS₂ geometry when $m_\alpha = 0$. We find that all of these masses satisfy the Breitenlohner-Freedman bound for all values of k and thus an instability of this specific kind is not present.

The outline of the remainder of the paper is as follows. In Sec. II, we briefly review the action and relevant black brane solution of massive gravity introduced in [21] and in Sec. III we present the equations of motion and on-shell action of linearized transverse fluctuations in this theory, emphasizing how the breaking of bulk diffeomorphism invariance can be explicitly seen to violate the field theory Ward identities due to momentum conservation. In Sec. IV we outline how hydrodynamics should be modified to account for the nonconservation of momentum, and compute the time scale of momentum relaxation by a study of the low energy transverse excitations of the dual field theory. Section V contains an analytic derivation of the low frequency conductivity of the zero temperature field theory state. In Sec. VI we begin an exploration of the stability of the solution of interest before finishing in Sec. VII with a summary of our results and some suggestions for future research. The appendix contains some simplifications of the fluctuation equations in the limit $m_\alpha = 0$, including a proof of the decoupling of the two ‘‘master fields’’ when $k = 0$.

II. MASSIVE GRAVITY AND ITS EQUILIBRIUM SOLUTION

The nonlinear theory of massive gravity that we will investigate couples the metric tensor $g_{\mu\nu}$ to a fixed reference metric $f_{\mu\nu}$, giving a mass to $g_{\mu\nu}$ and breaking diffeomorphism invariance. With the usual holographic motivation, we also include a negative cosmological constant and a minimally coupled Maxwell field A_μ in the action [21]

$$S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left(\mathcal{R} + \frac{6}{L^2} - \frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} + m^2 \{ \alpha [\mathcal{K}] + \beta ([\mathcal{K}]^2 - [\mathcal{K}^2]) \} \right), \quad (5)$$

where α, β are arbitrary dimensionless coupling constants, square brackets denote the trace $[\mathcal{K}] = \mathcal{K}^\mu_\mu$, indices are raised and lowered with the dynamical metric $g_{\mu\nu}$ and \mathcal{K} satisfies $\mathcal{K}^\mu_\alpha \mathcal{K}^\alpha_\nu = g^{\mu\alpha} f_{\alpha\nu}$ where the reference metric is chosen to be $f_{xx} = f_{yy} = F^2$ for a constant F , with all other components vanishing. Our coordinates are (t, x, y, r) where r is the holographic radial direction. This choice of reference metric clearly distinguishes the two spatial directions x, y from the temporal and radial directions t, r . This choice breaks the symmetries associated with reparametrizations

of the spatial x, y coordinates and will result in the dissipation of momentum (but not energy) in the dual field theory. It was shown in [21], following [29], that the Boulware-Deser ghost [22] may be absent in this theory (see [30] for a pedagogical review of theoretical aspects of massive gravity). We will not comment further upon the theory's nonlinear stability. Instead, we are primarily interested in whether any purported field theory dual of (5) is a sensible model for a condensed matter system which dissipates momentum. We will find, at the level of the two-point functions of the supposed field theory, that it is. We will henceforth assume the existence of a strongly-coupled field theory dual to (5), and use the usual holographic dictionary [31–35] to compute its properties.

The action (5) admits the black brane solution [21]

$$\begin{aligned}
 ds^2 &= \frac{L^2}{r^2} \left(-f(r)dt^2 + dx^2 + dy^2 + \frac{dr^2}{f(r)} \right), \\
 A_t(r) &= \mu \left(1 - \frac{r}{r_0} \right), \\
 f(r) &= 1 - r_0^2 m_\alpha^2 \frac{r}{r_0} - r_0^2 m_\beta^2 \frac{r^2}{r_0^2} \\
 &\quad - \left[1 - r_0^2 m_\alpha^2 - r_0^2 m_\beta^2 + \frac{1}{4} r_0^2 \mu^2 \right] \frac{r^3}{r_0^3} + \frac{1}{4} r_0^2 \mu^2 \frac{r^4}{r_0^4},
 \end{aligned} \tag{6}$$

where the radial coordinate r takes values between 0 (the boundary of the spacetime) and r_0 , the location of the black brane horizon, and the constants m_α^2 and m_β^2 (which may be either positive or negative) are related to the parameters of the action (5) via

$$m_\alpha^2 = -\frac{\alpha F L m^2}{2r_0}, \quad m_\beta^2 = -\beta F^2 m^2. \tag{7}$$

This solution is invariant under translations in the spatial field theory directions x, y (as well as in the temporal direction). As we will show in the following section, it is the lack of diffeomorphism invariance of the linearized fluctuations that will lead to momentum dissipation in the dual field theory. The diffeomorphism-breaking terms m_α^2 and m_β^2 appear as mass terms for the linearized metric perturbations.

The temperature T of the dual field theory state is given by

$$T = \frac{1}{4\pi r_0} \left[3 - 2(r_0 m_\alpha)^2 - (r_0 m_\beta)^2 - \frac{1}{4}(r_0 \mu)^2 \right], \tag{8}$$

and the chemical potential of the state is μ . A nonzero μ results in a nonzero density of charge in the field theory state. Further thermodynamic properties of this solution were studied in [21].¹ This state has four independent

¹Note added: A more careful study of the thermodynamics has subsequently been performed in [36].

dimensionful scales: $r_0, \mu, m_\alpha^2, m_\beta^2$. In the field theory, it is easier to think in terms of the more physically motivated scales $T, \mu, m_\alpha^2, m_\beta^2$, where we will later give some physical meaning to m_α^2 and m_β^2 in the field theory in terms of the momentum relaxation timescale. Near the horizon, the zero temperature black brane geometry has the form $\text{AdS}_2 \times \mathbb{R}^2$, which we will utilize in detail in Sec. V when computing the AC conductivity.

III. LINEARIZED FLUCTUATIONS AND WARD IDENTITIES

To determine the transport properties of the dual field theory, we must study fluctuations of the bulk fields $g_{\mu\nu}$ and A_μ around the background (6), which is translationally invariant in the (t, x, y) directions. It is simplest to work in Fourier space and thus we write

$$g_{\mu\nu}(r) \rightarrow g_{\mu\nu}(r) + \int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega t + ikx} h_{\mu\nu}(r, \omega, k), \tag{9}$$

$$A_\mu(r) \rightarrow A_\mu(r) + \int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega t + ikx} a_\mu(r, \omega, k).$$

Note that the Fourier transform defined here has the opposite sign in the exponent than the transform defined in [21]. For our purposes (which are to compute two-point Green's functions), it is sufficient to study the linearized fluctuations. As usual, the fluctuations can be classified as either odd or even under the transformation $y \rightarrow -y$. As both the action (5) and the background solution (6) are invariant under this transformation, the even fluctuations ($h_{tt}, h_{rx}, h_{xt}, h_{tt}, h_{xx}, h_{yy}, h_{rr}, a_r, a_x, a_t$) decouple from the odd ones ($h_{yx}, h_{yt}, h_{yr}, a_y$) at linearized order. We will refer to the fields which are odd/even under this transformation as transverse/longitudinal (with respect to the direction of k).

As in the usual gauge/gravity dictionary, we take A^μ to be dual to a conserved U(1) current J^a in the dual field theory and $g_{\mu\nu}$ to be dual to the energy-momentum tensor T^{ab} , which we will shortly show is no longer conserved.

A. Equations of motion with reduced gauge invariance

For the usual case of a theory with diffeomorphism invariance, these linearized fluctuations possess a gauge symmetry under infinitesimal coordinate transformations $\delta_\xi x^\mu = \xi^\mu(x)$ which act as

$$\begin{aligned}
 \delta_\xi h_{\mu\nu} &= -\bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu, \\
 \delta_\xi a_\mu &= -\xi^\alpha \bar{\nabla}_\alpha A_\mu - A_\alpha \bar{\nabla}_\mu \xi^\alpha,
 \end{aligned} \tag{10}$$

where the bar denotes a covariant derivative with respect to the background metric (6). In addition, there is an invariance under infinitesimal U(1) gauge transformations of the form

$$\delta_\Lambda a_\mu = a_\mu - \partial_\mu \Lambda, \quad \delta_\Lambda h_{\mu\nu} = 0. \tag{11}$$

These bulk gauge symmetries encode global symmetries of the dual field theory, as was noted during the original proposition of the AdS/CFT correspondence [37]. At the level of linearized fluctuations, it is possible to make this encoding very explicit by working with certain

gauge-invariant combinations of fluctuations. This then allows a clearer picture of the repercussions for the field theory of breaking the bulk diffeomorphism invariance.

From the action (5), the linearized equations of motion of the transverse fluctuations are

$$\frac{d}{dr} \left[\frac{1}{r^2} (h_t^y + i\omega h_r^y) + A'_t a_y \right] - \frac{k}{r^2 f} (k h_t^y + \omega h_x^y) - \frac{2}{r^2 f} \left(m_\beta^2 + m_\alpha^2 \frac{r_0}{r} \right) h_t^y = 0, \quad (12a)$$

$$\frac{d}{dr} \left[\frac{f}{r^2} (h_x^y - i k h_r^y) \right] + \frac{\omega}{r^2 f} (\omega h_x^y + k h_t^y) - \frac{r_0 m_\alpha^2}{r^3} h_x^y = 0, \quad (12b)$$

$$i\omega h_t^y + i k f h_x^y - \left[\omega^2 - k^2 f - 2f \left(m_\beta^2 + m_\alpha^2 \frac{r_0}{r} \right) \right] h_r^y + i\omega r^2 A'_t a_y = 0, \quad (12c)$$

$$\frac{d}{dr} [f a'_y] + A'_t (h_t^y + i\omega h_r^y) + \frac{1}{f} (\omega^2 - k^2 f) a_y = 0, \quad (12d)$$

where a prime denotes a derivative with respect to r and indices are raised and lowered using the background metric (6). These equations are equivalent to those given in [21] after setting $k = 0$, changing the sign of ω (due to the opposite sign used in the definition of the Fourier transform) and rescaling a_y by a factor of 2.

The main effect of the nondiffeomorphism-invariant terms in the action—apart from changing the background function $f(r)$ appearing in the fluctuation equations—is to produce explicit mass terms for the transverse components of $h_{\mu\nu}$ in the equations of motion above. That these mass terms break diffeomorphism invariance can be explicitly verified by noting that the equations of motion (12a)–(12c), with $m_\alpha^2, m_\beta^2 \neq 0$ are not invariant under the infinitesimal diffeomorphisms (10), under which the transverse fields transform as

$$\begin{aligned} \delta_\xi h_r^y &= -\frac{r^2}{L^2} \left(\xi'_y + \frac{2}{r} \xi_y \right), & \delta_\xi h_t^y &= \frac{i\omega r^2}{L^2} \xi_y, \\ \delta_\xi h_x^y &= -\frac{i k r^2}{L^2} \xi_y, & \delta_\xi a_y &= 0. \end{aligned} \quad (13)$$

As a consequence, we see that diffeomorphisms with $\xi_y \neq 0$ are no longer symmetries of the gravitational theory.

As one might expect, this loss of a gauge symmetry results in the creation of an extra dynamical degree of freedom. To best parametrize this, let us firstly recall how to write the massless theory in an explicitly diffeomorphism-invariant form [38]. To do this, we wish to select a set of diffeomorphism-invariant bulk fields to study, rather than the gauge-dependent fundamental fluctuations h_t^y etc. There is no unique way to select such a set from the bulk fields and their derivatives, but there is a natural choice to make. Fields $h_{\mu\nu}$ with both indices in the “field theory directions” (t, x, y) are dual to components of the conserved energy-momentum tensor T^{ab} of the strongly-coupled field theory, whereas the components of $h_{\mu\nu}$ with indices in the r -direction have no such direct field

theory interpretation. The natural choice is therefore to study gauge-invariant combinations of bulk fields which do not involve $h_{\mu r}$. With this restriction, the only diffeomorphism-invariant combinations of the transverse fields are

$$Z_1 = \frac{\omega}{k} h_x^y + h_t^y, \quad Z_2 = a_y, \quad (14)$$

or any linear combination of them. There is a simple procedure to obtain the equations of motion for these gauge-invariant combinations—one simply solves the “constraint” equation (12c) (in the massless limit) for the constraint field h_r^y , and then substitutes this solution into the remaining “dynamical” equations of motion (12a), (12b), and (12d) (in the massless limit). The resulting three equations are comprised of only two linearly independent equations which are the two coupled, dynamical (i.e. second order) differential equations for the gauge-invariant fields $Z_{1,2}$.

By repeating this procedure of substituting for the constraint field h_r^y in the on-shell action for the fluctuations, one can write it in a manifestly gauge-invariant form

$$S = \int_{r=0} \frac{d\omega dk}{(2\pi)^2} \left[Z_i^{(0)}(-\omega, -k) G_{ij}(\omega, k) Z_j^{(0)}(\omega, k) + \dots \right], \quad (15)$$

where the superscript (0) denotes the value of the field at the boundary, the ellipsis denotes contact terms (i.e. terms analytic in ω and k) and the explicit form of $G_{ij}(\omega, k)$ can be found in [39]. With this formalism, it is clear that the retarded Green’s functions of the dual operators, extracted via the usual holographic procedure, obey (up to contact terms) [38,40]

$$G_{T^{xy}T^{xy}}^R = \frac{\omega}{k} G_{T^{ty}T^{xy}}^R = \frac{\omega^2}{k^2} G_{T^{ty}T^{ty}}^R, \quad \text{etc.} \quad (16)$$

These are simply the Ward identities due to conservation of (transverse) energy-momentum in the field theory:

$\partial_a T^{ay} = 0$. This explicitly gauge-invariant formalism therefore manifestly encodes the corresponding global symmetries of the dual field theory (at the level of the two-point functions).

We will now repeat this procedure when $m_\alpha^2, m_\beta^2 \neq 0$. As before, it is simple to solve the constraint equation (12c) for h_r^y and substitute this solution into

the remaining dynamical equations of motion (12a), (12b), and (12d). As before, we have now eliminated h_r^y from our equations and will ascribe no meaning to this bulk field from the point of view of the dual field theory. In contrast to the massless case, the three resulting dynamical equations of motion are all linearly independent. They may be written

$$\frac{d}{dr} \left[\frac{f}{r^2} \frac{\omega(kh_t^{y'} + \omega h_x^{y'}) - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})h_x^{y'} + \omega k r^2 A'_t a_y}{\omega^2 - k^2 f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})} \right] + \frac{\omega}{r^2 f} (\omega h_x^y + k h_t^y) - \frac{r_0 m_\alpha^2}{r^3} h_x^y = 0, \quad (17a)$$

$$\frac{d}{dr} [f a_y'] - A'_t \frac{k f (k h_t^{y'} + \omega h_x^{y'}) + 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})h_t^{y'} + \omega^2 r^2 A'_t a_y}{\omega^2 - k^2 f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})} + \frac{1}{f} (\omega^2 - k^2 f) a_y = 0, \quad (17b)$$

$$\frac{d}{dr} \left[\frac{f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})(\omega h_t^{y'} + k f h_x^{y'} + \omega r^2 A'_t a_y)}{r^2 [\omega^2 - k^2 f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})]} \right] + \frac{\omega}{r^2 f} \left(m_\beta^2 + m_\alpha^2 \frac{r_0}{r} \right) h_t^y + \frac{k r_0 m_\alpha^2}{2 r^3} h_x^y = 0. \quad (17c)$$

Due to the breaking of diffeomorphism invariance, the fields h_x^y and h_t^y no longer appear in the previously gauge-invariant combination Z_1 but are now independent dynamical degrees of freedom. This means that the two point functions of T^{xy} and of T^{ty} in the dual field theory are now independent and that the Ward identities (16) are no longer satisfied. This, of course, is a consequence of the fact that (transverse) energy-momentum is no longer conserved in the dual field theory: $\partial_a T^{ay} \neq 0$. In total there are now three independent, dynamical transverse fields— h_t^y , h_x^y , and a_y —which are gauge-invariant with respect to the unbroken gauge symmetries (diffeomorphisms in the r, t directions and the U(1) gauge symmetry), which is one more than in the massless case as we previously anticipated. Note that in some situations (such as when computing the AC conductivity as we will do in Sec. V) it is more convenient to work with more involved bulk variables than the three fundamental fields themselves, but this does not affect the argument just outlined.

Similar effects—the generation of extra bulk dynamical degrees of freedom and the relaxation of field theory Ward identities due to conservation of energy-momentum—will occur in the longitudinal sector of the theory.

B. On-shell action

Near the boundary of the spacetime (6), each field h_t^y, h_x^y, a_y can be characterized by the two independent coefficients in its near-boundary power series expansion. Just as in the massless case, for h_t^y and h_x^y these are the coefficients of the terms of order r^0 and r^3 in said expansions, and for a_y they are the coefficients of the terms of order r^0 and r^1 . However, a nonzero value of m_α^2 does make one qualitative difference to these near-boundary expansions: it produces terms in the near-boundary expansions of all three fields which are logarithmic in r , in addition to the usual integer powers. Such terms usually arise in bulk theories with an odd number of dimensions and are related to the conformal anomaly of the dual field theory [41], but they appear here in an even-dimensional theory. These logarithms will not be important in what follows. In fact, all of the interesting qualitative features of our results can be found even when $m_\alpha^2 = 0$.

To compute observables in the dual field theory, we require the on-shell gravitational action. After eliminating h_r^y using the constraint equation (12c) and including the Gibbons-Hawking term in the action, this is given by

$$S = \frac{L^2}{2\kappa_4^2} \int_{r \rightarrow 0} \frac{d\omega dk}{(2\pi)^2} \left\{ \frac{f}{2r^2} \frac{1}{[\omega^2 - k^2 f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})]} \left[-(\omega h_x^y + k h_t^y)(\omega h_x^{y'} + k h_t^{y'}) - 2 \left(m_\beta^2 + m_\alpha^2 \frac{r_0}{r} \right) (h_t^y h_t^{y'} - f h_x^y h_x^{y'}) \right] - \frac{f}{2} a_y a_y' + \text{non-derivative terms} \right\}, \quad (18)$$

where a prime denotes a derivative with respect to r and the arguments of the first and second fluctuation in each pair are $(r, -\omega, -k)$ and (r, ω, k) , respectively. As expected from the breaking of diffeomorphism invariance just outlined, the fields in the on-shell action no longer appear in the diffeomorphism-invariant combinations (14).

The nonderivative terms, which we have not written explicitly, include the counterterms required to render the on-shell action finite. For the massless theory these counterterms are given, for example, in [42]. We assume that it is possible to write an analogous set of counterterms in the case with nonzero mass such that the on-shell action is

finite. These counterterms, and in fact all of the nonderivative terms in (18), will not affect any of the results we present because they contribute real contact terms (i.e. real terms analytic in ω , k) to the retarded Green's functions $G_{\mathcal{O}\mathcal{O}}^R(\omega, k)$ of operators in the dual theory. Most of the quantities which we compute holographically—poles of the Green's functions, and spectral functions—are not affected by such terms. The only quantity we compute which is sensitive to such terms is the imaginary part of the conductivity $\sigma(\omega)$ which will receive nonzero contributions from any $\sim a_y^2$ counterterms. However, since the part of the on-shell action (18) dependent upon a_y is already finite, we anticipate that there will be no counterterms of this form.

IV. “HYDRODYNAMICS,” MOMENTUM RELAXATION AND THE WALL OF STABILITY

A ubiquitous feature of translationally invariant media at nonzero temperatures is the applicability of hydrodynamics as an effective theory at small enough frequencies ω and momenta k . Hydrodynamics is a fluidlike limit based on the assumptions that the energy-momentum tensor and any global currents of a theory can be expressed in terms of a small number of slowly varying (with respect to a microscopic length scale l_{mfp}) macroscopic variables and their derivatives, and that the energy-momentum tensor and global currents are conserved [43,44]. These assumptions should be valid for a state in local thermal equilibrium. For simplicity, let us consider the case of a (2 + 1)-dimensional conformal fluid at nonzero temperature and with no conserved charges. In this case, the constitutive relation expressing the energy-momentum tensor in terms of macroscopic variables and their derivatives is [45]

$$T^{ab} = (\epsilon + P)u^a u^b + P g^{ab} - \eta(g^{ac} + u^a u^c) \times (g^{bd} + u^b u^d)(\partial_c u_d + \partial_d u_c - g_{cd}\partial_e u^e) + \dots, \quad (19)$$

where ϵ is the energy density in the fluid rest frame, P is the fluid pressure, u^a is the fluid three-velocity which obeys $u^a u_a = -1$, η is the shear viscosity, g_{ab} is the Minkowski metric and the ellipsis denotes terms which are higher order in spacetime derivatives and therefore suppressed at small frequencies and momenta by powers of ωl_{mfp} , $k l_{\text{mfp}} \ll 1$.

Hydrodynamics predicts the existence of two long-lived collective excitations in such a theory: a sound mode due to longitudinal fluid flow (i.e. u^i parallel to k , where i labels the spatial directions of the field theory) and a shear diffusion mode due to transverse fluid flow (u^i transverse to k). To show the existence of these modes, consider a field theory state which has been perturbed slightly from equilibrium such that

$$u_a = (-1, \delta u_x, \delta u_y), \quad \epsilon \rightarrow \epsilon + \delta\epsilon, \quad P \rightarrow P + \delta P. \quad (20)$$

Without loss of generality, we choose the momentum of the perturbation k to flow in the x -direction so that δu_a are functions of x and t only. The energy-momentum tensor of a fluid in this state is given by substituting this profile for u_a into (19). Demanding that T^{ab} is conserved for this configuration, we find (after Fourier transforming and keeping terms which are linear in the amplitude of perturbation) that this is only possible for perturbations satisfying specific dispersion relations. For longitudinal fluid flow ($\delta u_y = 0$), this is the dispersion relation of the sound mode and for transverse fluid flow ($\delta\epsilon = \delta P = \delta u_x = 0$), this is the dispersion relation of the shear diffusion mode

$$\omega = -i \frac{\eta}{\epsilon + P} k^2 + \dots, \quad (21)$$

where the ellipsis denotes higher order terms in k . Upon Fourier transforming to real space, this dispersion relation is that of a purely decaying mode (i.e. it has a vanishing propagation frequency) with decay rate $\Gamma = \eta k^2 / (\epsilon + P)$ or equivalently with a lifetime $\tau = [\eta k^2 / (\epsilon + P)]^{-1}$.

In the language of quantum field theory, these collective modes are realized as poles in the two-point functions of the longitudinal and transverse components of T^{ab} , respectively. These predictions of hydrodynamics (and many others) have been comprehensively verified for holographic theories with momentum conservation [46–52], and in particular they have been verified for the massless version of the theory [17,53–61] which we are studying. For holographic theories, a sufficient condition for the applicability of hydrodynamics is that we consider perturbations with $\omega, k \ll T$ (i.e. $l_{\text{mfp}} \sim T^{-1}$).

A. Modified hydrodynamics

After violating the conservation of momentum in our field theory by the inclusion of nonzero mass terms m_α^2 , $m_\beta^2 \neq 0$ in its gravitational dual, we no longer expect hydrodynamics to be the correct effective theory at low energies since energy-momentum is no longer conserved. In the following subsections, we will determine the effect of these nonzero mass terms on the shear diffusion mode of our field theory by computing the relevant Green's functions from its gravitational dual. First though, we will propose a modification of hydrodynamics which replicates these effects.

Our proposal is to consider a fluid with the same constitutive relation (19) as previously but, for small perturbations around the equilibrium state where the fluid is at rest, to replace the conservation equation with

$$\partial_a T^{at} = 0, \quad \partial_a T^{ai} = -(\epsilon + P)\tau_{\text{rel}}^{-1} u^i, \quad (22)$$

where τ_{rel} is a constant in spacetime. A similar modification was proposed in [17] in the context of impurity

scattering. What does this mean? Consider a spatially independent but time-dependent perturbation of the fluid. Then, according to Eq. (22), the energy density of the fluid is constant in time $\partial_t T^t = 0$, but the momentum density of the fluid is now time-dependent: $\partial_t T^{ti} \neq 0$. The fluid will lose momentum at a rate proportional to its velocity, with proportionality constant $(\epsilon + P)\tau_{\text{rel}}^{-1}$. Specifically, for a spatially-independent near-equilibrium fluid flow of the form (20), $T^{ti} = (\epsilon + P)u^i$ and hence our modification of the conservation law corresponds to

$$\partial_t T^t = 0, \quad \partial_t T^{ti} = -\tau_{\text{rel}}^{-1} T^{ti}. \quad (23)$$

We can therefore identify τ_{rel} as the momentum relaxation time scale in the system—i.e. the characteristic time scale over which the system loses momentum.

It is a simple task to repeat the previous calculations, with Eq. (22) replacing the conservation equation, to determine the transverse collective excitations of this effective theory. We find that the dispersion relation of the shear diffusion mode is altered to become

$$\omega = -i\left(\tau_{\text{rel}}^{-1} + \frac{\eta k^2}{\epsilon + P}\right) + \dots, \quad (24)$$

where the ellipsis represents higher order terms in k . In real space, this excitation is a purely decaying mode with decay rate $\Gamma = \tau_{\text{rel}}^{-1} + \eta k^2/(\epsilon + P)$. Our modification of the conservation equation to incorporate the loss of momentum results in the shear diffusion mode decay rate increasing by a constant equal to the inverse of the momentum relaxation time scale (or equivalently, equal to the rate of momentum relaxation). In other words, this excitation radiates its energy faster and thus lives for a shorter time.

We have thus far implicitly assumed that $\tau_{\text{rel}} > 0$. If in fact $\tau_{\text{rel}} < 0$, the fluid will actually *gain* momentum at a rate proportional to its velocity. Such a state is clearly unstable to any small excitations—for example, the decay rate of the shear diffusion mode will become negative and thus its amplitude will grow exponentially in time. We therefore demand that $\tau_{\text{rel}} \geq 0$ for stability.

What we have outlined is, like the usual theory of hydrodynamics, a phenomenological theory—we have no microscopic description of what is causing the momentum relaxation of the fluid described here. In a quasiparticle-based description, we can picture it as arising from scattering processes which do not conserve momentum but as usual in holography, a quasiparticle description is probably not an accurate description of the true dynamics of the strongly coupled field theory.

Furthermore, we emphasize that the modified version of hydrodynamics outlined above is not derived via the usual procedure in hydrodynamics (and in effective field theories more generally) of writing a constitutive relation containing all terms allowed by the symmetries present. While such an analysis would be worthwhile, instead we have just proposed the simplest modification which produces

the transverse collective excitation present in the strongly-interacting field theory dual to (5), which will be derived shortly. This simple modification is the dominant correction to hydrodynamics in this limit. In the field theory dual to (5), we find that the momentum relaxation rate is given by

$$\tau_{\text{rel}}^{-1} = \frac{s(m_\alpha^2 + m_\beta^2)}{2\pi(\epsilon + P)}, \quad (25)$$

and that the stability condition $\tau_{\text{rel}} \geq 0$ is equivalent to the “wall of stability” which was previously identified numerically for this theory. There are numerous other ways to test if this modified version of hydrodynamics really is an accurate description of the aforementioned field theory, the most obvious of which is to compute the dispersion relation of the (longitudinal) sound mode in both setups. We leave this to future work.

B. Zero density “hydrodynamics”

Gauge/gravity duality encodes the collective excitations of the field theory (i.e. the poles of the field theory two-point functions) as the quasinormal modes of the dual bulk gravitational fields. To determine the effect of the diffeomorphism-breaking mass terms m_α^2, m_β^2 on the shear diffusion mode, we therefore want to calculate the effects of these terms on the corresponding quasinormal mode of the bulk field fluctuations. The simplest case to consider is that of zero density ($\mu = 0$), where the fluctuations of the bulk gauge field decouple from those of the bulk metric. In the dual field theory, this means that small fluctuations of energy-momentum decouple from those of the U(1) current and therefore the modified hydrodynamics just introduced—which assumed that there were no such couplings—may apply.

In this decoupling limit, it is fluctuations of the transverse components of T^{ab} which support the shear diffusion mode. In this limit, the equations of motion of their dual gravitational fields are

$$\begin{aligned} \frac{d}{dr} \left[\frac{f}{r^2} \frac{\omega(kh_t^{y'} + \omega h_x^{y'}) - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})h_x^{y'}}{\omega^2 - k^2 f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})} \right] \\ + \frac{\omega}{r^2 f} (\omega h_x^y + kh_t^y) - \frac{r_0 m_\alpha^2}{r^3} h_x^y = 0, \end{aligned} \quad (26a)$$

$$\begin{aligned} \frac{d}{dr} \left[\frac{f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})(\omega h_t^{y'} + kf h_x^{y'})}{r^2 [\omega^2 - k^2 f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})]} \right] \\ + \frac{\omega}{r^2 f} \left(m_\beta^2 + m_\alpha^2 \frac{r_0}{r} \right) h_t^y + \frac{kr_0 m_\alpha^2}{2r^3} h_x^y = 0. \end{aligned} \quad (26b)$$

In this limit, the decoupled equation of motion (17b) for a_y is invariant under diffeomorphisms (13) and will not concern us here. The breaking of diffeomorphism invariance is communicated to it by its coupling to the metric fluctuations at nonzero density.

To compute the dispersion relation of the relevant quasinormal mode, we will divide the spacetime into two overlapping regions—an “inner region” close to the horizon at $r = r_0$ and an “outer region” which extends up to the boundary $r = 0$. We can then solve the equations of motion in each of these regions up to integration constants. A quasinormal mode must be purely infalling at the horizon: imposing this boundary condition fixes the integration constants of the inner region solutions (up to an overall normalization of each field). By demanding consistency of the inner and outer region solutions in the matching regions where they overlap, we can then fix the integration constants of the outer region solutions (up to an overall normalization of each field). To identify a quasinormal mode, we then determine the values of ω for which these outer region solutions are normalizable at the spacetime boundary.

1. Inner region solutions

The inner region is the region of spacetime close to the horizon defined by $r_0 - r \ll r_0$. Expanding the terms in the equations of motion (26a) and (26b) in this limit, we find that the fields h_t^y and h_x^y decouple, and both obey the equation (assuming that $T \neq 0$)

$$\frac{d}{dr}[(r_0 - r)h_t^{y'}(r)] + \frac{\omega^2}{f'(r_0)^2(r_0 - r)}h_t^y(r) = 0, \quad (27)$$

$$\begin{aligned} \frac{f}{r^2[\omega^2 - k^2f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})]} \left[\omega(\omega h_x^{y'} + k h_t^{y'}) - 2f\left(m_\beta^2 + m_\alpha^2 \frac{r_0}{r}\right)h_x^{y'} \right] &= \omega k c_1, \\ \frac{-f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})}{r^2[\omega^2 - k^2f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})]} (\omega h_t^{y'} + k f h_x^{y'}) &= \omega c_2, \end{aligned} \quad (30)$$

where c_1 and c_2 are integration constants that will be fixed shortly. It is simple to decouple the two fields and integrate to give the outer region solutions

$$\begin{aligned} h_x^y &= h_x^{y(0)} + \omega k c_1 \int_0^r d\hat{r} \frac{\hat{r}^2}{f(\hat{r})} + \omega k c_2 \int_0^r d\hat{r} \frac{\hat{r}^2}{f(\hat{r})[m_\beta^2 + m_\alpha^2 \frac{r_0}{\hat{r}}]}, \\ h_t^y &= h_t^{y(0)} + \frac{2}{3}c_2 r^3 - \frac{1}{3}c_1 k^2 r^3 - \omega^2 c_2 \int_0^r d\hat{r} \frac{\hat{r}^2}{f(\hat{r})[m_\beta^2 + m_\alpha^2 \frac{r_0}{\hat{r}}]}, \end{aligned} \quad (31)$$

where $h_x^{y(0)}$ and $h_t^{y(0)}$ denote the boundary values of the respective fields. These integrals can be done analytically but the results are very lengthy and will not be presented here. Near the boundary of the spacetime, the solutions take the form

$$\begin{aligned} h_x^y(r) &= h_x^{y(0)} + \frac{1}{3}\omega k c_1 r^3 + \dots, \\ h_t^y(r) &= h_t^{y(0)} + \frac{1}{3}(2c_2 - k^2 c_1)r^3 + \dots. \end{aligned} \quad (32)$$

From these expressions, we see that the quasinormal modes are given by the poles of the integration constants c_1 and c_2 .

and similarly for h_x^y . To compute the retarded Green's functions of the dual field theory, we need to impose ingoing boundary conditions on these fields at the horizon. After solving (27), we impose these ingoing boundary conditions to obtain the following solutions for the fields in the inner region

$$\begin{aligned} h_x^y &= a_0 \exp\left[-\frac{i\omega}{4\pi T} \log\left(\frac{r_0 - r}{r_0}\right)\right], \\ h_t^y &= b_0 \exp\left[-\frac{i\omega}{4\pi T} \log\left(\frac{r_0 - r}{r_0}\right)\right], \end{aligned} \quad (28)$$

where a_0 and b_0 are integration constants and we have used the expression (8) for the temperature T . This is the standard result for ingoing fields at a horizon of nonzero temperature.

2. Outer region solutions

We define the outer region of the spacetime by the limit

$$r^2 \ll \frac{f^2}{\omega^2 - k^2f - 2f(m_\beta^2 + m_\alpha^2 \frac{r_0}{r})}. \quad (29)$$

This region extends to the boundary of the spacetime at $r = 0$ but not all the way to the horizon. In this limit, the nonderivative terms in the equations of motion (26a) and (26b) are subleading and we can neglect them and thus trivially integrate each equation to yield the first order differential equations

These integration constants can be fixed by demanding that the outer solutions correspond to fields which are ingoing at the horizon, as we will now demonstrate.

3. Matching and Green's functions

Assuming that the momentum and mass terms are of the same order of magnitude as the frequency ($\omega \sim k \sim m_\alpha \sim m_\beta$), the inner and outer regions overlap in a region of spacetime given by

$$\left| \frac{\omega}{f'(r_0)} \right| \ll \frac{r_0 - r}{r_0} \ll 1. \quad (33)$$

Since $f'(r_0) \sim T$, this region of spacetime is of a non-zero size provided that $\omega, k, m_\alpha, m_\beta \ll T$. This is the hydrodynamic limit.

To impose ingoing boundary conditions on the outer solutions, we expand both the ingoing inner region solutions (28) and the outer region solutions (30) in this matching region. By demanding that these expansions are consistent, we fix c_1 and c_2 to be those corresponding to ingoing fields at the horizon. The inner region solutions in the matching region are given by expanding (28) in the limit $\omega \log \left(\frac{r_0 - r}{r_0} \right) \ll T$ to give

$$\begin{aligned} h_x^y &= a_0 \left[1 - \frac{i\omega}{4\pi T} \log \left(\frac{r_0 - r}{r_0} \right) + \dots \right], \\ h_t^y &= b_0 \left[1 - \frac{i\omega}{4\pi T} \log \left(\frac{r_0 - r}{r_0} \right) + \dots \right]. \end{aligned} \quad (34)$$

The outer region solutions in the matching region are found by expanding (30) in the limit $r_0 - r \ll r_0$. The constants c_1 and c_2 are fixed by demanding that the ratios of the

coefficients of the $(r_0 - r)^0$ and $\log \left(\frac{r_0 - r}{r_0} \right)$ terms in these expansions are the same as that for the inner region solutions in the matching region (34). This fixes c_1 and c_2 to be

$$\begin{aligned} c_1 &= \frac{3r_0(\omega h_x^{y(0)} + k h_t^{y(0)}) + 2i(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2) h_x^{y(0)} + \dots}{\kappa_4^2 r_0^3 [k^2 r_0^2 + 2(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2) - 3ir_0\omega + \dots]}, \\ c_2 &= \frac{-(m_\alpha^2 + m_\beta^2)(3h_t^{y(0)} - ikr_0 h_x^{y(0)}) + \dots}{r_0 [k^2 r_0^2 + 2(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2) - 3ir_0\omega + \dots]}, \end{aligned} \quad (35)$$

where the ellipses denote higher order terms in $\omega, k, m_\alpha, m_\beta$.

These constants exhibit a pole at a given $\omega(k, m_\alpha, m_\beta, T)$ which is the dispersion relation of the quasinormal mode of the dual gravitational solution as explained in the previous subsection. This coincides with the dispersion relation of the poles of the field theory retarded Green's functions. To compute these Green's functions, we evaluate the on-shell action (18) to give (at lowest order in $\omega, k, m_\alpha, m_\beta$)

$$\begin{aligned} S &= \frac{L^2}{4\kappa_4^2 r_0^3} \int \frac{d\omega dk}{(2\pi)^2} \frac{1}{k^2 r_0^2 + 2(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2) - 3ir_0\omega} \{ [-3r_0^2 k^2 - 6(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)] h_t^{y(0)} h_t^{y(0)} \\ &\quad - 3r_0^2 \omega k [h_x^{y(0)} h_t^{y(0)} + h_t^{y(0)} h_x^{y(0)}] + [-3r_0^2 \omega^2 - 2ir_0\omega(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)] h_x^{y(0)} h_x^{y(0)} \}, \end{aligned} \quad (36)$$

where $h_t^{y(0)} h_t^{y(0)}$ is shorthand for $h_t^{y(0)}(-\omega, -k) h_t^{y(0)}(\omega, k)$ and similarly for the other terms with superscript (0), and where we have neglected terms which are analytic in ω, k and thus produce contact terms in the dual field theory Green's functions.

Using the usual holographic prescription [62,63], we find that the retarded Green's functions of the dual field theory in the limit $\omega, k, m_\alpha, m_\beta \ll T$ are (up to contact terms)

$$\begin{aligned} G_{T^{xy}T^{xy}}^R(\omega, k) &= \frac{L^2}{2\kappa_4^2 r_0^3} \frac{3r_0^2 k^2 + 6(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)}{3ir_0\omega - k^2 r_0^2 - 2(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)}, & G_{T^{xy}T^{xy}}^R(\omega, k) &= \frac{L^2}{2\kappa_4^2 r_0^3} \frac{3r_0^2 \omega^2 + 2ir_0\omega(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)}{3ir_0\omega - k^2 r_0^2 - 2(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)}, \\ G_{T^{xy}T^{xy}}^R(\omega, k) &= G_{T^{xy}T^{xy}}^R(\omega, k) = \frac{L^2}{2\kappa_4^2 r_0^3} \frac{3r_0^2 \omega k}{3ir_0\omega - k^2 r_0^2 - 2(m_\alpha^2 r_0^2 + m_\beta^2 r_0^2)}. \end{aligned} \quad (37)$$

These Green's functions have two important features. First, they no longer satisfy the Ward identities due to momentum conservation (16), as we anticipated. Second, the Green's functions have a pole with dispersion relation

$$\omega = -i \frac{r_0 k^2}{3} - i \frac{2r_0}{3} (m_\alpha^2 + m_\beta^2) + \dots = -i \left(\frac{\eta k^2}{\epsilon + P} + \tau_{\text{rel}}^{-1} \right) + \dots, \quad (38)$$

where we have defined

$$\tau_{\text{rel}}^{-1} = \frac{s(m_\alpha^2 + m_\beta^2)}{2\pi(\epsilon + P)}, \quad (39)$$

and where we have used the expressions

$$\epsilon = \frac{L^2}{\kappa_4^2 r_0^3}, \quad P = \frac{\epsilon}{2}, \quad \eta = \frac{s}{4\pi} = \frac{L^2}{2\kappa_4^2 r_0^2}, \quad (40)$$

for the relevant thermodynamic and transport quantities when $\mu = m_\alpha = m_\beta = 0$. Although m_α and m_β are non-vanishing in our calculation, they are small with respect to

T and hence their effect on these thermodynamic quantities produces contributions to the dispersion relation which are subleading. Note that we have rewritten the $2r_0/3$ factor in τ_{rel}^{-1} in terms of thermodynamic quantities. Written in this way, we will shortly show that this relation holds also at nonzero densities.

This dispersion relation is precisely that expected from the modified version of hydrodynamics we described in Sec. IV A and we therefore interpret τ_{rel}^{-1} as the time scale of momentum relaxation in the field theory. Looking back

on the bulk equations of motion (26a) and (26b), one can see from the denominators of each equation that the effective momentum of each field near the horizon (where dissipation occurs) is $k_{\text{eff}}^2 = k^2 + 2(m_\alpha^2 + m_\beta^2)$ and so the contributions of m_α^2 and m_β^2 to the dispersion relation (38) are perhaps not too surprising.

Finally, we note that the mode (38) crosses to the upper half-plane and therefore is unstable when $\tau_{\text{rel}}^{-1} < 0$ (and $k = 0$). In terms of the parameters of the original action, this condition for stability may be written

$$\frac{F^2 m^2 s}{(\epsilon + P)} \left(\beta + \frac{\alpha L}{2r_0 F} \right) \leq 0, \quad (41)$$

which coincides precisely with the ‘‘wall of stability’’ identified numerically in [21]. As indicated previously, this is a homogeneous (i.e. $k = 0$) instability which occurs because the system begins to gain (rather than lose) momentum at a constant rate such that the momentum of the system grows exponentially in time.

C. Nonzero density ‘‘hydrodynamics’’ and the Drude model

Let us now generalize our previous discussion of modified hydrodynamics to a theory with a conserved U(1) charge. When such a charge is present, we must supplement the previous hydrodynamic equations by a constitutive relation for the U(1) current J^a and the conservation equation $\partial_a J^a = 0$. We will consider a current with the usual constitutive relation of that of a conformal theory

$$J^a = \rho u^a + \dots, \quad (42)$$

where ρ is the charge density and the ellipsis denotes terms of higher order in spacetime derivatives. To compute the collective excitations, we can consider the near-equilibrium configuration (20) with $\rho \rightarrow \rho + \delta\rho$ also. The equations of motion for J^a and T^{ab} are now coupled but for transverse fluid flow (only $\delta u_y \neq 0$), the conservation equation for J^a is identically zero and thus the collective mode dispersion relation is (24), the same as before.

There is one difference with respect to the zero charge density case which is that the collective shear diffusion mode, which is an eigenmode with $\delta u_y \neq 0$, now results in non-trivial flow of current J^y in the transverse direction [as can be seen from Eq. (42)] in addition to a nontrivial flow of transverse energy-momentum. In the hydrodynamic limit of a field theoretical system, this results in the two-point functions of J^y , T^{ty} and T^{xy} all having a pole with the dispersion relation of this collective mode.

This case of a system with a nonvanishing conserved U(1) charge is relevant for the strongly coupled field theory dual to (5) when $\mu \neq 0$.² Our modified version of

²Note that although we have broken bulk diffeomorphism invariance, we have not broken bulk U(1) gauge invariance and so the U(1) current in the dual field theory is still conserved.

hydrodynamics predicts a shear diffusion mode with dispersion relation

$$\begin{aligned} \omega &= -i \frac{\eta}{\epsilon + P} k^2 - i \tau_{\text{rel}}^{-1} + \dots, \\ &= -i \frac{r_0}{3(1 + \frac{1}{4} \mu^2 r_0^2)} [k^2 + 2(m_\alpha^2 + m_\beta^2)] + \dots, \end{aligned} \quad (43)$$

where we have used the result (39) for τ_{rel}^{-1} and the following thermodynamic quantities valid in the $m_\alpha^2 = m_\beta^2 = 0$ limit

$$\epsilon = \frac{L^2}{\kappa_4^2 r_0^3} \left(1 + \frac{1}{4} \mu^2 r_0^2 \right), \quad P = \frac{\epsilon}{2}, \quad \eta = \frac{s}{4\pi} = \frac{L^2}{2\kappa_4^2 r_0^2}. \quad (44)$$

As before, the corrections to these due to nonzero m_α^2 , m_β^2 will produce subleading terms in the dispersion relations.

To check the validity of this result at nonzero densities, we wish to compute the poles of the retarded Green’s functions of J^y , T^{ty} and T^{xy} by determining the quasinormal modes of the dual bulk fields. The extra coupling between a_y and the metric fluctuations h_t^y , h_x^y when $\mu \neq 0$ makes this a complicated task. We therefore computed the poles of these Green’s functions numerically by integrating the equations of motion (17a)–(17c) from the horizon (where we applied ingoing boundary conditions) to the boundary for various ω and searching for the existence of a quasinormal mode using the procedure described in [63]. Note that the Green’s functions all share a common set of poles as their dual bulk fields are coupled. Working in the limit ω , k , m_α , $m_\beta \ll T$, we found a quasinormal mode with dispersion relation (43) for all values of μ that we studied. A sample of these results are shown in Fig. 1.

In summary, we have therefore shown that our modified version of hydrodynamics—in which we accounted for the relaxation of momentum over a single characteristic time

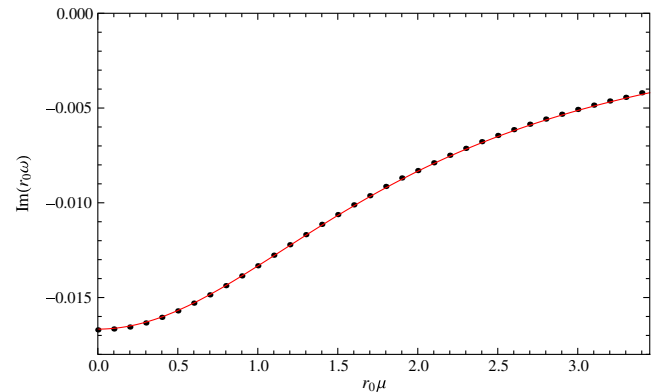


FIG. 1 (color online). Dependence of the imaginary part of the shear diffusion mode upon the field theory chemical potential μ , for $r_0^2 m_\alpha^2 = r_0^2 m_\beta^2 = 0.01$. The real part (not shown) is zero. The black dots are the results from numerically integrating the equations of motion, and the solid red line is the expression (43).

scale τ_{rel} —is an accurate description of the field theory dual to the massive gravity theory (5) with $\mu \neq 0$ in the hydrodynamic limit $\omega, k, m_\alpha, m_\beta \ll T$. As in the $\mu = 0$ case, there is an instability when $\tau_{\text{rel}} < 0$ due to momentum absorption. This coincides with the “wall of stability” found numerically in [21].

In fact, the charge transport in this modified hydrodynamic theory is essentially the same as that of the simple Drude model. At low enough energies and momenta we can approximate the full two-point function of J^y by just the contribution from the shear diffusion pole. Using the Kubo formula, we can then determine the conductivity

$$\sigma(\omega) \equiv \frac{i}{\omega} G_{J^y J^y}^R(\omega, k=0) \sim \frac{1}{\tau_{\text{rel}}^{-1} - i\omega} \sim \frac{1}{1 - i\omega\tau_{\text{rel}}}, \quad (45)$$

where we do not have an analytic expression for the numerator and we have not included corrections that are higher order in ω . This has the same form as the conductivity computed using the Drude model in which the current is composed of charged, classical particles which, in the absence of external forces, undergo collisions with ions with a mean free time τ_{rel} (see [21] for a brief review of this). This result clearly applies more generally than in the specific microscopic model of Drude, as we have shown here, and relies only on the hydrodynamic structure just outlined.

V. THE ZERO TEMPERATURE CONDUCTIVITY: BEYOND THE DRUDE MODEL

As mentioned in the introduction, it is of interest to study non-Drude behavior of the conductivity in light of unexplained experimental measurements of the conductivity in the normal phase of certain high- T_c superconductors [28]. These interesting measurements were performed at low temperatures $T < \omega$ outside the hydrodynamic limit, and furthermore it was shown in [21] that the field theory dual to (5) exhibits such non-Drude behavior at low temperatures. In this section we will compute the conductivity of this theory when $T = 0$ and determine analytically the corrections to the simple Drude model. We will first briefly recall the important features of the conductivity for the massless case where momentum is conserved.

The conductivity can be determined from the retarded Green’s function of the transverse current via the Kubo formula

$$\sigma(\omega) \equiv \frac{i}{\omega} G_{J^y J^y}^R(\omega, k=0). \quad (46)$$

When $m_\alpha^2 = m_\beta^2 = 0$, the leading behavior of the conductivity at small ω (neglecting the delta function which is present due to translational invariance) is $\text{Re}(\sigma) \propto \omega^2$, $\text{Im}(\sigma) \propto \omega^{-1}$. The scaling of $\text{Im}(\sigma)$ simply indicates that there must be a delta function in $\text{Re}(\sigma)$, due to the

Kramers-Kronig relation [34]. This delta function has a simple physical interpretation: because momentum is conserved, the charges will accelerate indefinitely under an applied field and thus the DC conductivity is infinite. Neglecting this delta function, the scaling of $\text{Re}(\sigma)$ is controlled by the dimension of a scalar operator in the 1-dimensional CFT that governs the theory at low energies as we will now briefly review.

In the massless case [64], the bulk field a_y dual to J^y decouples from the bulk metric fluctuations when $k = 0$ and can therefore be treated independently. Near the horizon, this field behaves like a scalar field in AdS_2 with mass $(mL_2)^2 = 2$ where $L_2 = L/\sqrt{6}$ is the radius of the near-horizon AdS_2 geometry. Utilizing the AdS/CFT correspondence, we then expect that the low-energy dissipative properties of the two-point function of J^y are controlled by the two-point function of a scalar operator of dimension $\Delta = 2$ in the CFT_1 dual to the near-horizon AdS_2 geometry. This expectation is borne out by a careful calculation of the conductivity, the result of which is [64]

$$\begin{aligned} \sigma(\omega) &\propto \frac{i}{\omega} \frac{1 + t_0 \mathcal{G}_2(\omega) + \dots}{1 + t_1 \mathcal{G}_2(\omega) + \dots} \rightarrow \text{Re}(\sigma) \propto \omega^2 + \dots, \\ \text{Im}(\sigma) &\propto \omega^{-1} + \dots, \end{aligned} \quad (47)$$

where $\mathcal{G}_2(\omega) \sim i\omega^3$ is the Green’s function of a scalar operator of dimension 2 in the CFT_1 dual to the near-horizon AdS_2 region, t_i are ω -independent constants and ellipses denote higher order terms in a small ω expansion. As previously stated, the power of the leading term in $\text{Re}(\sigma)$ at small ω is controlled by the CFT_1 dual to the near-horizon geometry. A similar power law holds for many other holographic theories [65,66].

When $m_\alpha^2, m_\beta^2 \neq 0$, our background solution (6) still possesses an AdS_2 near-horizon geometry. However, the bulk fields a_y and h_t^y are now coupled. This is a consequence of the extra dynamical bulk degrees of freedom present when the graviton has a mass. The result is that, at low energies, the current J^y couples to two scalar operators in the CFT_1 —one with $\Delta = 2$ (as in the massless case), and one with $\Delta = 1$. In the limit $\omega \rightarrow 0$, the coupling to the scalar operator with $\Delta = 1$ will dominate and thus $\text{Re}(\sigma) \propto \omega^{-1} \mathcal{G}_1(\omega) = \omega^0$, where $\mathcal{G}_1(\omega) \sim i\omega$ is the two-point function of a scalar operator with $\Delta = 1$ in the CFT_1 . This results in a finite DC conductivity at zero temperature.

A. The zero momentum master fields

To compute the conductivity analytically, we will restrict ourselves to the case $m_\alpha = 0$. In this limit, we can write the $k = 0$ equations of motion involving a_y as two decoupled equations for two “master fields,” for all values of r . These decoupled “master fields” are given by

$$\varphi_\pm(r) = a_y + \gamma_\pm \left[r A'_t a_y + \frac{1}{r} (h_t^y)' + i\omega h_r^y \right], \quad (48)$$

where

$$\gamma_{\pm} = -\frac{3}{4m_{\beta}^2\mu r_0^2} \left[\left(1 - r_0^2 m_{\beta}^2 + \frac{1}{4} r_0^2 \mu^2 \right) \pm \sqrt{\left(1 - r_0^2 m_{\beta}^2 + \frac{1}{4} r_0^2 \mu^2 \right)^2 + \frac{8}{9} m_{\beta}^2 \mu^2 r_0^4} \right], \quad (49)$$

and they obey the equations of motion

$$\frac{d}{dr} [f(r)\varphi'_{\pm}(r)] + \varphi_{\pm}(r) \left\{ \frac{\omega^2}{f} - \frac{r^2 \mu^2}{r_0^2} - \frac{2m_{\beta}^2 \mu \gamma_{\pm} r}{r_0} \right\} = 0. \quad (50)$$

These equations are derived in the Appendix. In the $T = 0$ limit,

$$r_0 \mu = 2\sqrt{3 - (r_0 m_{\beta})^2}, \quad (51)$$

and thus the equations of motion simplify to

$$\frac{d}{dr} [f\varphi'_{\pm}(r)] + \varphi_{\pm}(r) \left\{ \frac{\omega^2}{f} + \frac{3r}{r_0^3} \left[2 - r_0^2 m_{\beta}^2 \pm \left(2 - \frac{1}{3} r_0^2 m_{\beta}^2 \right) \right] - \frac{4(3 - r_0^2 m_{\beta}^2)r^2}{r_0^4} \right\} = 0, \quad (52)$$

for the fields

$$\varphi_{\pm} = a_y - \frac{3[2 - r_0^2 m_{\beta}^2 \pm (2 - \frac{1}{3} r_0^2 m_{\beta}^2)]}{4r_0 m_{\beta}^2 \sqrt{3 - r_0^2 m_{\beta}^2}} \left\{ r A'_y a_y + \frac{1}{r} (h_i^{y'} + i\omega h_r^y) \right\}. \quad (53)$$

At $T = 0$ our background solution has one dimensionless parameter, $m_{\beta}^2 r_0^2$, which is the ratio of the graviton mass to the field theory chemical potential, or equivalently $\mu \tau_{\text{rel}}$ in the dual field theory as we will see shortly. We restrict to $m_{\beta}^2 r_0^2 \leq 3$ so that μ , given in (51), is always real.

To compute the conductivity, we will use a similar strategy to that used for the massless theory [64]. We divide the spacetime into inner and outer regions, solve the equations in each region, and match them in an overlapping region to fix the integration constants of the outer region solutions such that the fields are ingoing at the horizon. We can then read off the values of the fields at the boundary and, from the on-shell action, compute the Green's functions and hence the conductivity from the Kubo formula (46).

B. Solutions in the inner AdS₂ region and beyond

When $T = 0$, the geometry (6) has a near-horizon AdS₂ region which we expect will control the low energy, dissipative properties of the dual field theory. To see this near-horizon region at the level of the equations of motion, we change the radial coordinate to [64,67]

$$\zeta = \frac{\omega r_0}{(6 - m_{\beta}^2 r_0^2)} \frac{r_0}{(r_0 - r)}. \quad (54)$$

After writing the equations of motion (52) in this new radial coordinate, we can expand at small ω (keeping ζ fixed) to give

$$\begin{aligned} \varphi_+''(\zeta) + \varphi_+'(\zeta) \left[\frac{2(4 - r_0^2 m_{\beta}^2) r_0 \omega}{(6 - r_0^2 m_{\beta}^2)^2 \zeta^2} \right] + \varphi_+(\zeta) \left[1 + \frac{4[3 - r_0^2 m_{\beta}^2 + \zeta^2(4 - r_0^2 m_{\beta}^2)] r_0 \omega}{(6 - r_0^2 m_{\beta}^2)^2 \zeta^3} \right] &= 0, \\ \varphi_-''(\zeta) + \varphi_-'(\zeta) \left[\frac{2(4 - r_0^2 m_{\beta}^2) r_0 \omega}{(6 - r_0^2 m_{\beta}^2)^2 \zeta^2} \right] + \varphi_-(\zeta) \left[1 - \frac{2}{\zeta^2} + \frac{2(4 - r_0^2 m_{\beta}^2)(1 + 2\zeta^2) r_0 \omega}{(6 - r_0^2 m_{\beta}^2)^2 \zeta^3} \right] &= 0, \end{aligned} \quad (55)$$

where we have neglected terms of order ω^2 and higher. This expansion is valid very close to the horizon of the spacetime with ζ , which is roughly $r_0^2 \omega / (r_0 - r)$, kept fixed. At very low energies in this inner region, we can set $\omega = 0$ in (55) and find that (after rescaling ζ by a factor of ω) φ_+ and φ_- obey the equations of motion of scalar fields of mass $(mL_2)^2 = 0, 2$ in an AdS₂ spacetime with radius $L_2 = L/\sqrt{6 - m_{\beta}^2 r_0^2}$. Therefore, with a nonzero

mass m_{β} , there is still an emergent CFT₁ controlling the low-energy dissipative dynamics of the field theory.

However, the mass term m_{β} does have one important consequence for these dynamics. The dissipative dynamics of J^y are now controlled by *two* operators in this CFT₁—one of dimension $\Delta = 1$ and one of dimension $\Delta = 2$ —since both of φ_{\pm} depend upon a_y . It is the coupling to this dimension 1 operator that will produce a finite DC conductivity.

We can solve the inner region equation (55) perturbatively in ω . After imposing ingoing boundary conditions on the solutions at the horizon $\zeta \rightarrow \infty$, these are

$$\begin{aligned}\varphi_+(\zeta) &= e^{i\zeta} \left\{ a_0 + a_1 \omega + \frac{2a_0 r_0 \omega}{\zeta(6 - m_\beta^2 r_0^2)^2} [-(3 - m_\beta^2 r_0^2) + i(4 - m_\beta^2 r_0^2) \zeta \log \zeta] \right\} \\ &\quad + e^{-i\zeta} \frac{4ia_0 \omega r_0 (3 - r_0^2 m_\beta^2)}{(6 - m_\beta^2 r_0^2)^2} [\text{Ei}(2i\zeta) - i\pi] + O(\omega^2), \\ \varphi_-(\zeta) &= b_0 \left(1 + \frac{i}{\zeta} \right) e^{i\zeta} + O(\omega),\end{aligned}\tag{56}$$

where $\text{Ei}(z)$ is the exponential integral [68] and a_0, a_1 and b_0 are integration constants. Because φ_- is dual to an operator of higher dimension than φ_+ , its contribution to the conductivity at low ω will be subleading, and therefore we only require it to leading order in ω as shown above.

Note that the solutions (56), which include corrections of order ω^1 , go beyond the simple form of scalar fields in AdS_2 which are valid to order ω^0 . These solutions are sensitive not just to the AdS_2 geometry infinitesimally

close to the horizon but also to corrections to this geometry encapsulated by the order ω^1 corrections.

C. Solutions in the outer region

We can define an outer region of the spacetime, which includes the boundary at $r = 0$, by the inequality $r^2 \omega^2 \ll f^2$. In this limit, the equations of motion (52) become

$$\frac{d}{dr} [f \varphi'_\pm(r)] + \varphi_\pm(r) \left\{ \frac{3r}{r_0^3} \left[2 - r_0^2 m_\beta^2 \pm \left(2 - \frac{1}{3} r_0^2 m_\beta^2 \right) \right] - \frac{4(3 - r_0^2 m_\beta^2) r^2}{r_0^4} \right\} = 0,\tag{57}$$

and can be solved analytically to give

$$\begin{aligned}\varphi_+ &= \left[1 + \frac{2(3 - m_\beta^2 r_0^2) r}{m_\beta^2 r_0^3} \right] \left\{ \varphi_+^{(0)} + \theta_+ \left[\frac{r_0(6 - m_\beta^2 r_0^2)}{r_0 - r} - \frac{8r_0(6 - m_\beta^2 r_0^2)(3 - m_\beta^2 r_0^2)^2}{(2 - m_\beta^2 r_0^2)[2(3 - m_\beta^2 r_0^2)r + m_\beta^2 r_0^3]} \right. \right. \\ &\quad - (6 - m_\beta^2 r_0^2) + \frac{8(6 - m_\beta^2 r_0^2)(3 - m_\beta^2 r_0^2)^2}{m_\beta^2 r_0^2(2 - m_\beta^2 r_0^2)} - 2(10 - 3m_\beta^2 r_0^2) \log \left(\frac{r_0 - r}{r_0} \right) \\ &\quad \left. + \frac{-m_\beta^6 r_0^6 - 3m_\beta^4 r_0^4 + 48m_\beta^2 r_0^2 - 92}{(2 - m_\beta^2 r_0^2)^{\frac{3}{2}}} \left\{ \tan^{-1} \left(\frac{(3 - m_\beta^2 r_0^2)r + r_0}{r_0 \sqrt{2 - m_\beta^2 r_0^2}} \right) - \tan^{-1} \left(\frac{1}{\sqrt{2 - m_\beta^2 r_0^2}} \right) \right\} \right. \\ &\quad \left. + (10 - 3m_\beta^2 r_0^2) \log \left[\frac{(3 - m_\beta^2 r_0^2)r^2 + 2rr_0 + r_0^2}{r_0^2} \right] \right\},\end{aligned}\tag{58}$$

and

$$\begin{aligned}\varphi_- &= \frac{r_0 - r}{r_0} \left\{ \varphi_-^{(0)} + \theta_- \left[r_0(6 - m_\beta^2 r_0^2)(13m_\beta^4 r_0^4 - 111m_\beta^2 r_0^2 + 246) + 12r_0(-m_\beta^6 r_0^6 + 11m_\beta^4 r_0^4 - 42m_\beta^2 r_0^2 + 56) \log \left(\frac{r_0 - r}{r_0} \right) \right. \right. \\ &\quad - 6r_0(-m_\beta^6 r_0^6 + 11m_\beta^4 r_0^4 - 42m_\beta^2 r_0^2 + 56) \log \left[\frac{(3 - m_\beta^2 r_0^2)r^2 + 2rr_0 + r_0^2}{r_0^2} \right] \\ &\quad \left. - \frac{3r_0(m_\beta^8 r_0^8 - 10m_\beta^6 r_0^6 + 37m_\beta^4 r_0^4 - 68m_\beta^2 r_0^2 + 68)}{\sqrt{2 - m_\beta^2 r_0^2}} \left[\tan^{-1} \left(\frac{(3 - m_\beta^2 r_0^2)r + r_0}{r_0 \sqrt{2 - m_\beta^2 r_0^2}} \right) - \tan^{-1} \left(\frac{1}{\sqrt{2 - m_\beta^2 r_0^2}} \right) \right] \right] \right\} \\ &\quad + \frac{1}{(r_0 - r)^2} \{ -(6 - m_\beta^2 r_0^2) r_0 \theta_- [3r^2(3m_\beta^4 r_0^4 - 23m_\beta^2 r_0^2 + 46) - 3rr_0(7m_\beta^4 r_0^4 - 56m_\beta^2 r_0^2 + 116) \\ &\quad + r_0^2(13m_\beta^4 r_0^4 - 111m_\beta^2 r_0^2 + 246)] \},\end{aligned}\tag{59}$$

where $\varphi_\pm^{(0)}$ are the values of φ_\pm at $r = 0$ and θ_\pm are integration constants.

D. Matching and the conductivity

To fix the integration constants θ_{\pm} such that the fields are ingoing at the horizon, we expand the outer region solutions (58) and (59) near the horizon $r_0 - r \ll r_0$, and the ingoing inner region solutions (56) far from the horizon $\zeta \rightarrow 0$. In this “matching region” given by

$$\frac{\omega r_0}{6 - m_{\beta}^2 r_0^2} \ll \frac{r_0 - r}{r_0} \ll 1, \quad (60)$$

we can fix θ_{\pm} by matching the terms of order $(r - r_0)^{-1}$, $\log(\frac{r_0 - r}{r_0})$ and $(r_0 - r)^0$ in φ_+ and the terms of order $(r_0 - r)^{-2}$ and $(r_0 - r)^{-1}$ in φ_- . This yields

$$\begin{aligned} \theta_+(\omega) &= \frac{i\omega r_0 \varphi_+^{(0)} + \dots}{(6 - m_{\beta}^2 r_0^2)^2} \left[1 + \kappa_1 i\omega \log\left(\frac{\omega r_0}{6 - m_{\beta}^2 r_0^2}\right) + \kappa_2 \omega + \kappa_3 i\omega + \dots \right]^{-1}, \\ \theta_-(\omega) &= -\frac{i\omega^3 r_0^2 \varphi_-^{(0)}}{3(6 - m_{\beta}^2 r_0^2)^6} [1 + \dots], \end{aligned} \quad (61)$$

where the ellipses denote higher order terms in ω and the constants κ_i in $\theta_+(\omega)$ are

$$\begin{aligned} \kappa_1 &= \frac{8r_0(3 - m_{\beta}^2 r_0^2)}{(6 - m_{\beta}^2 r_0^2)^2}, \quad \kappa_2 = \frac{4\pi r_0(3 - m_{\beta}^2 r_0^2)}{(6 - m_{\beta}^2 r_0^2)^2}, \\ \kappa_3 &= \frac{r_0}{(6 - m_{\beta}^2 r_0^2)^2} \left[(3 - m_{\beta}^2 r_0^2)(8\gamma_E + 8\log 2 - 9) - \frac{16(3 - m_{\beta}^2 r_0^2)^3}{m_{\beta}^2 r_0^2(2 - m_{\beta}^2 r_0^2)} + 3(4 - m_{\beta}^2 r_0^2) - \frac{-m_{\beta}^6 r_0^6 - 3m_{\beta}^4 r_0^4 + 48m_{\beta}^2 r_0^2 - 92}{(2 - m_{\beta}^2 r_0^2)^{\frac{3}{2}}} \right. \\ &\quad \left. \times \left\{ \tan^{-1}\left(\frac{4 - m_{\beta}^2 r_0^2}{\sqrt{2 - m_{\beta}^2 r_0^2}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{2 - m_{\beta}^2 r_0^2}}\right) \right\} - (10 - 3m_{\beta}^2 r_0^2) \log(6 - m_{\beta}^2 r_0^2) \right], \end{aligned} \quad (62)$$

with γ_E the Euler-Mascheroni constant. $\theta_{\pm}(\omega)$ are expansions in the quantity $\omega r_0 / (6 - m_{\beta}^2 r_0^2)^2$, which, from (60), must be small for the matching to be accurate.

To determine the two-point function of J^y , we need to know the value of the on-shell action for a bulk field configuration satisfying $a_y^{(0)} = 1$, $h_t^{y(0)} = 0$ where the superscript (0) denotes the boundary value of the corresponding field. By comparing the expansions of the solutions for φ_{\pm} [given in (58) and (59)] around $r = 0$ with the expression obtained by expanding the fundamental fields a_y and h_t^y around $r = 0$ and substituting into the definition (53) of φ_{\pm} , we can solve for the coefficients in the expansions of a_y and h_t^y in terms of the coefficients in the expansions of φ_{\pm} . After substituting the expansions of these fundamental fields into the on-shell action (18), we find that the on-shell action for a field configuration with $h_t^{y(0)} = 0$ is

$$\begin{aligned} S &= \frac{L^2}{4\kappa_4^2} \int \frac{d\omega}{2\pi} \frac{(6 - m_{\beta}^2 r_0^2)^3}{r_0} \\ &\quad \times \left[-\frac{1}{m_{\beta}^2 r_0^2} \frac{\theta_+(\omega)}{\varphi_+^{(0)}} + 6r_0(3 - m_{\beta}^2 r_0^2) \frac{\theta_-(\omega)}{\varphi_-^{(0)}} \right] \\ &\quad \times a_y^{(0)}(-\omega) a_y^{(0)}(\omega). \end{aligned} \quad (63)$$

At this point we will ignore the contribution from $\theta_-(\omega)$ —it is suppressed by a power of ω^2 with respect to $\theta_+(\omega)$ in the small ω limit where our calculation is valid. Since we have neglected terms of this order in the expansion of $\theta_+(\omega)$,

we should neglect the contribution of $\theta_-(\omega)$ for consistency. The leading order scaling behavior of $\theta_{\pm}(\omega)$ is, in fact, fixed by the dimension of the operator dual to φ_{\pm} in the near-horizon CFT₁. At leading order,

$$\theta_+(\omega) \sim \mathcal{G}_1(\omega) = i\omega, \quad \theta_-(\omega) \sim \mathcal{G}_2(\omega) = i\omega^3, \quad (64)$$

where $\mathcal{G}_{\Delta}(\omega)$ is the retarded Green’s function of a scalar operator of dimension Δ in the near-horizon CFT₁. It is now clear that due to the overlap of J^y with the $\Delta = 1$ operator in the near-horizon CFT, the conductivity $\sigma(\omega)$ will have a real part that scales as $\mathcal{G}_1(\omega)/\omega \sim \omega^0$ as compared to the $m_{\beta} = 0$ case where it has a delta function plus a part that scales as $\sim \mathcal{G}_2(\omega)/\omega \sim \omega^2$. This is the finite DC conductivity we anticipated due to the non-conservation of momentum.

To see this explicitly, we can determine the retarded Green’s function of J^y from (63) via the usual method and then, using the Kubo formula (46), the conductivity is

$$\begin{aligned} \sigma(\omega) &= \frac{L^2}{2\kappa_4^2} \frac{(6 - m_{\beta}^2 r_0^2)}{m_{\beta}^2 r_0^2} \\ &\quad \times \frac{1 + \dots}{1 + \kappa_1 i\omega \log\left(\frac{\omega r_0}{6 - m_{\beta}^2 r_0^2}\right) + \kappa_2 \omega + \kappa_3 i\omega + \dots} \\ &= \frac{\sigma_{\text{DC}} + \dots}{1 + \kappa_1 i\omega \log\left(\frac{\omega r_0}{6 - m_{\beta}^2 r_0^2}\right) + \kappa_2 \omega + \kappa_3 i\omega + \dots}, \end{aligned} \quad (65)$$

where the ellipses denote terms of order ω^2 and higher, the constants κ_i were given previously in (62), and the DC conductivity of the state is given by

$$\sigma_{\text{DC}} \equiv \sigma(\omega = 0) = \frac{L^2}{2\kappa_4} \frac{(6 - m_\beta^2 r_0^2)}{m_\beta^2 r_0^2}. \quad (66)$$

The first thing to note is that in the massless limit $m_\beta \rightarrow 0$, σ_{DC} diverges. This divergence indicates the formation of a delta function $\delta(\omega)$ in the conductivity when momentum is conserved, as we expected.

If we expand in the small mass limit $m_\beta^2 r_0^2 \ll \omega r_0 \ll 1$, the presence of the $\sim(m_\beta r_0)^{-2}$ term in κ_3 leads to

$$\sigma(\omega) = \frac{\sigma_{\text{DC}}}{1 - i\omega\tau_{\text{rel}}}, \quad (67)$$

where

$$\tau_{\text{rel}} = \frac{6}{m_\beta^2 r_0}. \quad (68)$$

This is precisely the Drude model result. This should not be too surprising, as the small mass limit is the limit where momentum conservation is only weakly violated (or equivalently, where the momentum relaxation time $\mu\tau_{\text{rel}}$ is very large). Our full expression (65) for the conductivity therefore incorporates both the Drude model, and the leading corrections to it. We will shortly explore these corrections in more detail, although unfortunately we do not have any microscopic field theoretic explanation of them.

We note that the momentum relaxation time τ_{rel} for small masses (68)—i.e. the relaxation time in the limit $\mu\tau_{\text{rel}} \gg 1$ —coincides with the naive $T = 0$ limit of the hydrodynamic momentum relaxation time (49). We expect that, as for the shear diffusion constant in the massless theory [39], the limit of applicability of the hydrodynamic result (39) is in fact $\tau_{\text{rel}}(T + \mu) \gg 1$. As in the hydrodynamic limit, the system absorbs momentum and is therefore unstable unless $\tau_{\text{rel}} \geq 0$ i.e. $m_\beta^2 \geq 0$.

Finally, we note that the DC resistivity ρ_{DC} is nonzero at $T = 0$. This was also noted by taking the low temperature limit in [21]. It was shown in [4] (and confirmed in [1]) that for a theory with a near-horizon AdS_2 geometry that is coupled to a lattice with wave vector k_L , the DC resistivity has a power law dependence of the form $\rho_{\text{DC}} \sim T^{2\nu(k_L)-1}$ for a given function $\nu(k_L)$. In the limit $k_L \rightarrow 0$, this becomes $\rho_{\text{DC}} \sim T^0$ which is what we have found here, although the prefactor of the power law T -dependence in ρ_{DC} vanishes as $k_L \rightarrow 0$. In [69], it was shown for a class of holographic models which have a near-horizon geometry conformal to AdS_2 that the DC resistivity due to scattering from random impurities is of the form $\rho_{\text{DC}} \sim T^{2\nu(0)-1}$ (although things are more complicated when the near-horizon geometry is AdS_2 itself [4]). It would be interesting to further explore this potential connection with massive gravity.

E. Corrections to the Drude model and the scaling region

To illustrate the effect of the non-Drude corrections to the conductivity, we show in Fig. 2 the exact conductivity $\sigma(\omega)$ for two different values of $m_\beta^2 r_0^2$, as computed numerically by integrating the equations of motion, and both the Drude conductivity (67) and the more accurate result (65) which includes corrections to the Drude formula. This figure illustrates a few important points. First, at small ω , the expression (65) gives a good approximation to the exact conductivity. For small values of $m_\beta^2 r_0^2$, there is a well-defined Drude-like peak near the origin of $\text{Re}(\sigma)$ and as $m_\beta^2 r_0^2$ increases, this peak spreads out, transferring its spectral weight to higher ω . In the extreme limit $m_\beta^2 r_0^2 = 3$ (which corresponds to $r_0\mu = 0$), the peak completely disappears and the low energy $\text{Re}(\sigma)$ is constant. This is simply because the current decouples from the momentum in this limit and so can relax quickly. The range of ω over which our result (65) is valid decreases as $r_0^2 m_\beta^2$ increases (i.e. as the Drude peak spreads out more). This is expected because our result is an expansion in powers of small $\omega r_0 / (6 - m_\beta^2 r_0^2)^2$ and therefore as $m_\beta^2 r_0^2$ increases, our result (65) is valid over a smaller range of frequencies.

Second, the consequences of the Drude peak spreading out as $m_\beta^2 r_0^2$ increases are that $\text{Re}(\sigma)$ is enhanced at larger ω while $\text{Im}(\sigma)$ is suppressed, relative to the Drude result. The corrections to the Drude model encapsulated in (65) capture these deviations to some extent, but not completely. For the Drude model at large frequencies $\omega\tau_{\text{rel}} \gg 1$, the phase of the conductivity $\arg \sigma = 90^\circ$. The corrections to the Drude model just described reduce this phase, with the amount of reduction increasing with $r_0^2 m_\beta^2$.

The phase is a particularly interesting quantity because certain high- T_c superconductors in the normal phase have been found to exhibit scaling law conductivities of the form [28]

$$\sigma(\omega) = K\omega^{\gamma-2} e^{i\frac{\pi}{2}(2-\gamma)}, \quad (69)$$

with $\gamma \approx 1.35$ and a constant K , in the range of frequencies corresponding to $\omega \gg T$, τ_{rel}^{-1} (up to some high energy cutoff which for us is $\sim\mu$). This scaling law, which is taken as a signal of underlying quantum criticality and is currently unexplained, manifests itself in two clearly observable ways: scaling behavior of the magnitude of the conductivity $|\sigma| = K\omega^{\gamma-2}$ and a constant phase $\arg \sigma = (2 - \gamma)90^\circ$.

In [21], it was shown that at small temperatures the conductivity of the theory dual to (5) behaves like a scaling law with a constant term $|\sigma| = c + K\omega^{\gamma-2}$, and an approximately constant phase. Similar behavior with a robust value of $\gamma \approx 1.35$ was seen in the holographic lattice models [1,2]. In Fig. 3 we plot $d \log(|\sigma| - c) / d \log \omega$ (which should be a constant equal to $\gamma - 2$ if σ is described by the expression above) and the phase of σ .

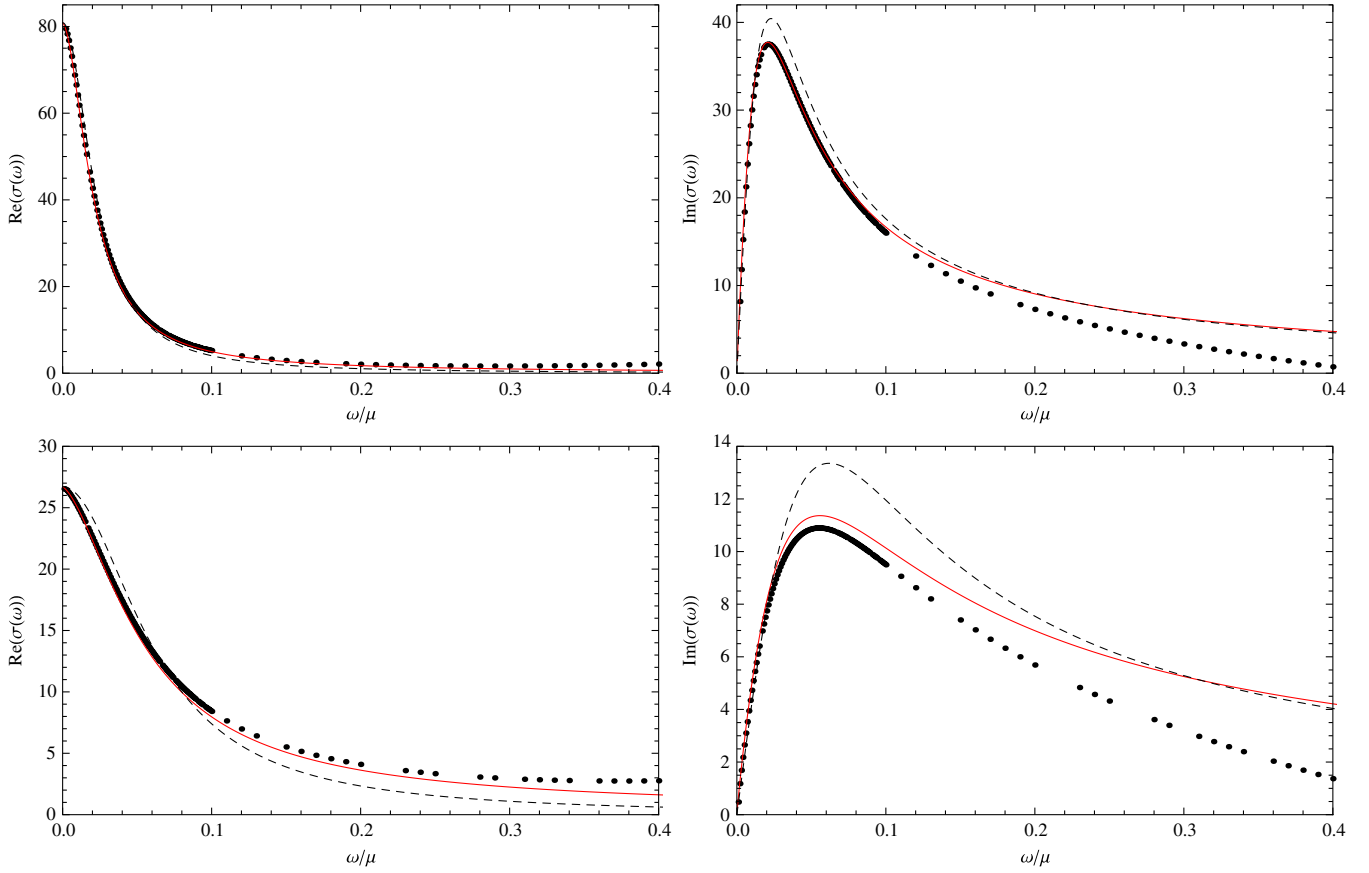


FIG. 2 (color online). The real and imaginary parts of the conductivity σ for $m_\beta^2 r_0^2 \approx 0.44$ (top) and $m_\beta^2 r_0^2 \approx 1.04$ (bottom). The black dots are the exact numerical results, the dashed black line is the Drude conductivity (67) and the solid red line is the result (65) which includes corrections to the simple Drude formula. σ is plotted in units of $L^2/(4\kappa_4^2 r_0 \mu)$.

By increasing $r_0 m_\beta$ from zero to its maximum value, one can obtain values of γ between 1 and 2, and phases between 90° and 0° . We have chosen $r_0^2 m_\beta^2 \approx 0.44$ such that γ , as extracted from the logarithmic plot, is ≈ 1.35 . The phase is not constant, but is relatively close to $(2 - \gamma)90^\circ \approx 60^\circ$ over a large range of ω .

One goal of our analytic computation of the low ω conductivity was to better understand these results. Specifically, it was to determine whether there really is a scaling behavior similar to (69) (with an extra constant term), and whether this can be understood as arising from the semilocal quantum critical state dual to the

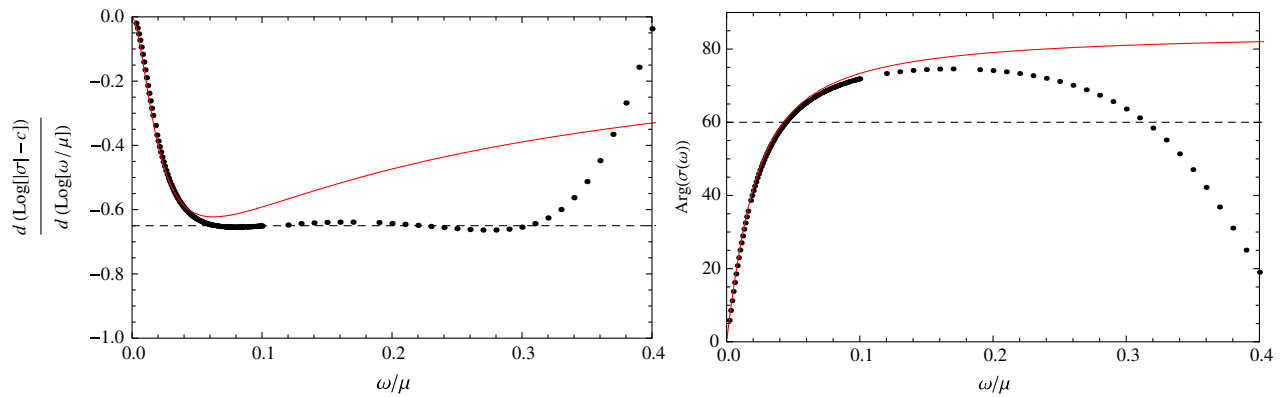


FIG. 3 (color online). The potential scaling exponent (left) as described in the main text with $c = -8.9$, and phase (right), of the conductivity σ when $m_\beta^2 r_0^2 \approx 0.44$. Black dots show the exact numerical results, the solid red line is the analytic expression (65) and the dashed black lines denote the exponent and phase found experimentally in [28]. Both c and σ are given in units of $L^2/(4\kappa_4^2 r_0 \mu)$.

near-horizon AdS₂ geometry. Although our analytic result (65) does not show any sign of this scaling behavior, it can be seen in Fig. 3 that our analytic results are unfortunately not close enough to the exact result in the regime of interest to conclusively determine whether there is such a power law or not. However, we assert that any potential scaling behavior of this type in the conductivity cannot be simply traced back to the presence of a near-horizon AdS₂ geometry, as our calculation captures accurately the effects of this region of the spacetime upon $\sigma(\omega)$, and does not produce a power law. Interpreting the radial direction as an energy scale in the usual way, it seems that the region of ω over which there is potentially a scaling law is far enough from $\omega = 0$ that the observables in this region should be strongly affected by regions of r which are not infinitesimally close to r_0 , i.e. by the geometry not next to the horizon. For this reason, it would be interesting to study geometries in which the spacetime has a scaling symmetry over a range of intermediate values of r between the horizon and boundary. Similar conclusions were reached for models incorporating an explicit bulk lattice in [3].

VI. A PRELIMINARY SEARCH FOR SPATIALLY MODULATED INSTABILITIES

As we have emphasized, the equilibrium properties of the field theory state we are studying exhibit spatial translational invariance. One may worry that the preferred equilibrium solution may be a state without such a symmetry. In this section we will search for a specific kind of

instability of the solution (6) with $m_\alpha = 0$, which would result in the formation of spatially modulated expectation values for J^y , T^{ly} or T^{yx} in the dual field theory, and find that no such instability occurs.

The existence of instabilities leading to spatially modulated equilibrium states are well-known in holographic models of states of matter at finite density. One source of such an instability is when the momentum-dependent mass of a field violates the Breitenlohner-Freedman (BF) bound [70] of the near-horizon geometry over a range of nonzero momenta [71–74]. This type of instability is often associated with Chern-Simons or axionic terms in the bulk action, neither of which are present here, but it was recently shown in [75] that instabilities of this kind are also possible when such terms are not present. The violation of the near-horizon BF bound is a well-studied topic in holography [76–81] and typically leads to a BKT-type transition in the dual field theory.

It is relatively straightforward to identify any such spatially modulated instabilities in our theory. We will work at $T = 0$ and examine the masses of the transverse fields that we have studied throughout this work. As in the previous section, for simplicity we will restrict ourselves to the case $m_\alpha = 0$, where we may write the equations of motion in the relatively simple form (A3a)–(A3c) for the three independent fields $\varphi_{1,2,3}$ given in (A1). To access the near-horizon AdS₂ region, we change coordinates to (54) and expand the resulting equations of motion as power series in ω . At lowest order in ω , the equations of motion are

$$\varphi_1'' + \varphi_1 - \frac{r_0^2(2m_\beta^2 + k^2)}{(6 - r_0^2m_\beta^2)\zeta^2} \varphi_1 - \frac{2\sqrt{3 - r_0^2m_\beta^2}(2m_\beta^2 + k^2)}{(6 - r_0^2m_\beta^2)\zeta^2} \varphi_2 = 0, \quad (70a)$$

$$\varphi_2'' + \varphi_2 - \frac{r_0^2k^2 + 4(3 - r_0^2m_\beta^2)}{(6 - r_0^2m_\beta^2)\zeta^2} \varphi_2 - \frac{2r_0^2\sqrt{3 - r_0^2m_\beta^2}}{(6 - r_0^2m_\beta^2)\zeta^2} \varphi_1 = 0, \quad (70b)$$

$$\varphi_3'' + \varphi_3 = 0. \quad (70c)$$

These equations may be diagonalized by changing variables to

$$\begin{aligned} \phi_1 &= \varphi_1 + \frac{1}{2r_0^2\sqrt{3 - r_0^2m_\beta^2}} \left[3(2 - r_0^2m_\beta^2) + \sqrt{(6 - r_0^2m_\beta^2)^2 + 4k^2r_0^2(3 - r_0^2m_\beta^2)} \right] \varphi_2, \\ \phi_2 &= \varphi_1 + \frac{1}{2r_0^2\sqrt{3 - r_0^2m_\beta^2}} \left[3(2 - r_0^2m_\beta^2) - \sqrt{(6 - r_0^2m_\beta^2)^2 + 4k^2r_0^2(3 - r_0^2m_\beta^2)} \right] \varphi_2, \quad \phi_3 = \varphi_3, \end{aligned} \quad (71)$$

whose equations of motion are simply the equations of motion of scalar fields in AdS₂ (after rescaling ζ by a factor of ω) with masses

$$\begin{aligned} (m_1L_2)^2 &= \frac{1}{(6 - r_0^2m_\beta^2)} \left[6 - r_0^2m_\beta^2 + k^2r_0^2 + \sqrt{(6 - r_0^2m_\beta^2)^2 + 4k^2r_0^2(3 - r_0^2m_\beta^2)} \right], \\ (m_2L_2)^2 &= \frac{1}{(6 - r_0^2m_\beta^2)} \left[6 - r_0^2m_\beta^2 + k^2r_0^2 - \sqrt{(6 - r_0^2m_\beta^2)^2 + 4k^2r_0^2(3 - r_0^2m_\beta^2)} \right], \quad (m_3L_2)^2 = 0. \end{aligned} \quad (72)$$

For real k , and $0 \leq m_\beta^2 r_0^2 \leq 3$ (the range of values of $m_\beta^2 r_0^2$ where the previously-identified $k = 0$ instability does not occur), it is clear that $(m_1 L_2)^2$, $(m_3 L_2)^2 \geq 0$. The BF bound in AdS₂ is $(m L_2)^2 > -1/4$ and so there is no instability due to $\phi_{1,3}$. For ϕ_2 , this is not so obvious. If we consider a fixed bulk graviton mass m_β , $(m_2 L_2)^2(k)$ is an even function of k and goes from 0 at $k = 0$ to $+\infty$ as $k \rightarrow \infty$. By taking a partial derivative with respect to k , its turning points k_* are given by

$$k_*^2 r_0^2 = 0, \quad \text{and} \quad k_*^2 r_0^2 = -\frac{3r_0^2 m_\beta^2 (4 - r_0^2 m_\beta^2)}{4(3 - r_0^2 m_\beta^2)}. \quad (73)$$

For real k , and $0 \leq m_\beta^2 r_0^2 < 3$, this second equation has no solutions and thus the only turning point is at $k = 0$. In other words, $(m_2 L_2)^2$ is an even function of k which monotonically increases from 0 to ∞ as k goes from 0 to ∞ . Therefore it never violates the AdS₂ BF bound.

A real lattice of course has spatial modulation of equilibrium observables (such as charge density and energy density) corresponding to longitudinal fields, and it is in the longitudinal sector that the type of nonzero k instability discussed here is present in [75]. It is therefore clearly important to repeat the above calculation for the longitudinal fields, but this is beyond the scope of this paper.

We emphasize here that although we have ruled out instabilities of the nature just described, there are of course other possible sources of $k \neq 0$ instabilities in the transverse fields that we have not investigated—for example, those associated with the dynamics of the fields outside the near-horizon region and also those of states with $m_\alpha \neq 0$ —as well as instabilities in the longitudinal fields as mentioned above. To identify instabilities arising outside the near-horizon region, one should compute the quasinormal mode spectrum and determine if there are any modes with positive imaginary part [e.g. by applying Leavers method [82] to the equations of motion (A3a)–(A3c)]. There is also the possibility of the existence of first order transitions to more stable states, which will not be identified even by this quasinormal mode analysis.

VII. DISCUSSION

In summary, we have investigated the effect of the relaxation of momentum upon various observables in the field theory holographically dual to the massive gravity action (5). In the hydrodynamic limit, the dominant effect of this relaxation can be incorporated into the low energy effective theory by a modification of the conservation equation of energy-momentum to (22), such that momentum is no longer conserved. We computed the characteristic time scale over which momentum relaxes in this limit (39) and identified the “wall of stability” found in [21] with the point at which this time scale vanishes. We also computed analytically the AC conductivity of this theory at zero temperature (65), including corrections to the simple

Drude model result. Finally, we made a preliminary study of the possible existence of $k \neq 0$ instabilities, and didn’t find any due to near-horizon BF bound violation by the transverse fields.

Our results are broadly what one would expect of a physical system which dissipates momentum and further justify the use of the massive gravity action (5) as a holographic model of such a system. Having calculated the momentum relaxation time scale and conductivity in terms of the graviton masses, we would like to know what microscopic field theoretical quantities (if any) these graviton masses correspond to. To do this, it is clearly important to better understand in what sense we can discuss a possible field theory dual of (5). For example, can the theory (5) be derived as the long-distance effective theory of a conventional holographic system (i.e. one with massless gravitons) in which translational invariance is broken by a well-understood microscopic mechanism such as by coupling to impurities [17,18] or by the presence of a lattice [1–4]? We note that one can generate a mass term for gravitons in a holographic setup by coupling two CFTs such that their dual theory comprises two AdS spaces with a shared boundary [83–88]. In this setup, the energy-momentum tensor of each individual CFT is not conserved due to interactions with the other.

Additionally, there are a few more straightforward calculations which would be of interest. One is to examine the hydrodynamic excitations in the longitudinal sector to see if they agree with the modified version of hydrodynamics we have proposed, and a second is to perform a more complete stability analysis of the solution along the lines outlined in Sec. VI, with particular attention paid to the possible existence of spatially inhomogeneous instabilities.

ACKNOWLEDGMENTS

I would like to thank Sean Hartnoll, Elias Kiritsis, Andrei Parnachev, Koenraad Schalm, David Vegh and Jan Zaanen for helpful discussions and correspondence. I am especially grateful to A. Parnachev and D. Vegh for their comments on a draft of this manuscript. This work was supported by a VIDI innovative research grant from NWO, The Netherlands Organisation for Scientific Research.

APPENDIX: SIMPLIFIED EQUATIONS OF MOTION FOR $m_\alpha = 0$

For solving the equations (12a)–(12d) in the $m_\alpha = 0$ limit, it is convenient to use the variables

$$\begin{aligned} \varphi_1 &= \frac{1}{r^2} (h_t^y + i\omega h_r^y) + A'_y a_y, & \varphi_2 &= a_y, \\ \varphi_3 &= \frac{f}{r^2} (h_x^y - ik h_r^y), \end{aligned} \quad (A1)$$

which are linear combinations of the fundamental fields and their derivatives. As was the case when we used the

fundamental fields h_t^y , h_x^y , and a_y , there are three linearly-independent dynamical variables φ_i . In the $m_\alpha = 0$ limit, the equations of motion can be written in terms of these variables as

$$\frac{d}{dr}[r^2 f \varphi_1'] - k(kh_t^y + \omega h_x^y) - 2m_\beta^2 h_t^y = 0, \quad (\text{A2a})$$

$$\frac{d}{dr}[f \varphi_2'] + r^2 A_t' \varphi_1 + \left[\frac{1}{f}(\omega^2 - k^2 f) - r^2 A_t'^2 \right] \varphi_2 = 0, \quad (\text{A2b})$$

$$\frac{d}{dr}[r^2 f \varphi_3'] + \omega(kh_t^y + \omega h_x^y) = 0, \quad (\text{A2c})$$

in addition to the constraint (12c). Using this constraint and the definitions of the fields φ_i in (A1), these become

$$\frac{d}{dr}[r^2 f \varphi_1'] + \frac{r^2}{f}[\omega^2 - k^2 f - 2f m_\beta^2] \varphi_1 + r^2 A_t'(2m_\beta^2 + k^2) \varphi_2 = 0, \quad (\text{A3a})$$

$$\frac{d}{dr}[f \varphi_2'] + r^2 A_t' \varphi_1 + \left[\frac{1}{f}(\omega^2 - k^2 f) - r^2 A_t'^2 \right] \varphi_2 = 0, \quad (\text{A3b})$$

$$\frac{d}{dr}[r^2 f \varphi_3'] + \frac{\omega^2 r^2}{f} \varphi_3 + \omega k r^2 \varphi_1 - \omega k r^2 A_t' \varphi_2 = 0. \quad (\text{A3c})$$

1. Decoupling of fields when $k = 0$

In the $k = 0$ limit, the field φ_3 decouples from φ_1 and φ_2 . This is analogous to the decoupling of h_x^y from h_t^y and a_y in the same limit. By defining the field $\tilde{\varphi}_1 \equiv r \varphi_1$ and partially expanding the derivative in the φ_1 equation of motion (A3a), we find that the coupled equations are

$$\frac{d}{dr}[f \tilde{\varphi}_1'] + \left(\frac{\omega^2}{f} - 2m_\beta^2 - \frac{f'}{r} \right) \tilde{\varphi}_1 + 2m_\beta^2 r A_t' \varphi_2 = 0, \quad (\text{A4a})$$

$$\frac{d}{dr}[f \varphi_2'] + \left(\frac{\omega^2}{f} - r^2 A_t'^2 \right) \varphi_2 + r A_t' \tilde{\varphi}_1 = 0. \quad (\text{A4b})$$

Taking the linear combination $\gamma \times (77a) + (77b)$, we find that the equations of motion take the form

$$\frac{d}{dr}[f \varphi_2'] + \gamma \frac{d}{dr}[f \tilde{\varphi}_1'] + \left(\frac{\omega^2}{f} - r^2 A_t'^2 + 2\gamma m_\beta^2 r A_t' \right) (\varphi_2 + \gamma \tilde{\varphi}_1) = 0, \quad (\text{A5})$$

provided that γ satisfies the quadratic equation

$$-2m_\beta^2 r A_t' \gamma^2 + \gamma \left(r^2 A_t'^2 - 2m_\beta^2 - \frac{f'}{r} \right) + r A_t' = 0. \quad (\text{A6})$$

The solutions to this quadratic equation, given by

$$\gamma_\pm = -\frac{3}{4m_\beta^2 r_0^2 \mu} \left[\left(1 - r_0^2 m_\beta^2 + \frac{1}{4} r_0^2 \mu^2 \right) \pm \sqrt{\left(1 - r_0^2 m_\beta^2 + \frac{1}{4} r_0^2 \mu^2 \right)^2 + \frac{8}{9} m_\beta^2 \mu^2 r_0^4} \right], \quad (\text{A7})$$

are in fact independent of r , and thus we can write the equations of motion (A5) in the decoupled form (50) for the variables (48). These are somewhat analogous to the decoupled Kodama-Ishibashi variables for the massless theory with $k \neq 0$ [89].

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