



Universiteit
Leiden
The Netherlands

The spherical albedo of a planet covered with a homogeneous cloud layer

Hulst, H.C. van de

Citation

Hulst, H. C. van de. (1974). The spherical albedo of a planet covered with a homogeneous cloud layer. *Astronomy And Astrophysics*, 35, 209-214. Retrieved from <https://hdl.handle.net/1887/7166>

Version: Not Applicable (or Unknown)

License: [Leiden University Non-exclusive license](#)

Downloaded from: <https://hdl.handle.net/1887/7166>

Note: To cite this publication please use the final published version (if applicable).

The Spherical Albedo of a Planet Covered with a Homogeneous Cloud Layer

H. C. van de Hulst

Sterrewacht, Leiden

Received February 15, 1974

Summary. The similarity rules valid for reflection against planetary cloud layers permit us to transform the set of parameters a (albedo for single scattering), g (asymmetry factor for single scattering) and b (optical thickness) in such a manner that the numerical results remain about the same. Three forms of such rules, which coincide for nearly-conservative scattering, are explained and a preference is expressed for one of them, in which the parameter $s = [(1-a)/(1-ag)]^{\frac{1}{2}}$ is kept invariant. The numerical test on the spherical albedo

A^* of a homogeneous semi-infinite atmosphere with Henyey-Greenstein scattering confirms that A^* may be expressed as a simple function of s alone, with deviations reaching at most 0.002 for any set of parameters (a, g) . Such a close similarity cannot be found for the plane albedo.

Key words: cloud layer — planet — similarity relations — diffuse reflection — spherical albedo

1. Posing the Similarity Problem

Although many accurate methods to solve problems of multiple light scattering in a planetary atmosphere are known, the need for quick, approximate methods remains. Among these the use of similarity relations takes a special place, because this method does not solve the problem from scratch but relates it to a problem with different parameters already solved. This is particularly useful if it permits with sufficient accuracy the reduction of a problem with arbitrary anisotropic phase function to one with isotropic scattering.

The purpose of this paper is twofold. We shall express a preference among three nearly equivalent ways in which the similarity relations for non-conservative scattering may be formulated. And we shall support this by a numerical test on the spherical albedo for a homogeneous semi-infinite atmosphere, yielding for this quantity a universal expression correct to 0.002 for any phase function.

Consider diffuse reflection from a homogeneous part of a planet's surface, or from the top surface of its cloud cover. For simplicity we assume absence of polarization. The diffuse reflection function $R(\mu, \mu_0, \varphi)$ is a function of μ_0 , the cosine of the angle of incidence, μ , the cosine of the angle of emergence, and φ , the azimuth difference between the plane of incidence and emergence. We write the azimuth average, or zero-order coefficient in the azimuthal Fourier expansion, as

$$R(\mu, \mu_0) = (2\pi)^{-1} \int_0^{2\pi} R(\mu, \mu_0, \varphi) d\varphi. \quad (1)$$

This function is symmetric in μ and μ_0 . We choose the normalization so that $R=1$ for a white Lambert surface. This is now the customary choice, although many authors in the tradition of Chandrasekhar (1950) prefer to use $S(\mu, \mu_0) = 4\mu_0\mu R(\mu, \mu_0)$. Two integral expressions are often needed. They are:

plane albedo

$$r(\mu) = \int_0^1 R(\mu, \mu_0) 2\mu_0 d\mu_0, \quad (2)$$

spherical albedo

$$A^* = \int_0^1 r(\mu) 2\mu d\mu = \int_0^1 \int_0^1 R(\mu, \mu_0) 4\mu\mu_0 d\mu d\mu_0. \quad (3)$$

Here the explanation requires a side step. It is customary to introduce the spherical albedo as a property of the entire planet, not of a plane surface or atmosphere. It is defined as the fraction of the incident solar radiation (in a chosen wave-length interval) diffusely reflected by the planet and its atmosphere over all directions. One possible way of computing it is first, at a given angle sun-planet-earth, to integrate over the illuminated and visible part of the disk in order to find the phase function of the planet for that angle. And by integrating over that angle we can then find the spherical albedo A^* . The equations may be found in standard texts (Horak, 1950; Harris, 1961).

However, this is not the full story. The texts just cited leave the distinct impression that the route to compute

A^* via the planetary phase function is the best or only method. The route via the plane albedo, expressed by Eq. (3) above, is generally simpler and fully equivalent. This fact is clearly stated in the literature (Russell, 1916; Ambartsumyan, 1958; Sobolev, 1974) but it may be useful to prove it once again as follows.

If we could divide up the sun's disk in many pieces and distribute them evenly around the entire sky, the fraction of the incident radiation, returned by the planet, would be the same. But each part of the planet's surface (or top surface of cloud cover) would then be exposed to uniform radiation from all directions. The fraction of energy returned by a plane surface in such uniform illumination is just the expression (3) above. If this is the fraction returned by each part of the planet, and the surface or cloud cover is homogeneous, then Eq. (3) also expresses the spherical albedo of the entire planet.

All of the functions and quantities defined above depend on the nature of the surface, or cloud cover. We now specify a homogeneous atmosphere without reflecting bottom surface. It is characterized by

b = total optical thickness,

a = albedo for single scattering,

$\Phi(\cos \alpha)$ = the complete phase function for single scattering.

The phase function may be expanded in the usual manner in Legendre functions, starting with $\omega_0 = 1$. The specification is complete if all coefficients $\omega_n (n \geq 1)$ of this expansion are known. Most important, undoubtedly, is the coefficient ω_1 , which permits us to consider

$g = \langle \cos \alpha \rangle = \frac{1}{3} \omega_1$ = asymmetry factor of the phase function

as the third essential parameter of the problem.

The similarity problem now is posed as follows. If the three parameters b , a , g are given, is it then possible to find another set of parameters b' , a' , g' , for which the reflection properties of the atmosphere are about the same? In particular, if we wish to use available results for isotropic scattering ($g' = 0$), what is then the most appropriate choice of b' and a' ?

2. Three Forms and Preferred Choice

The aim is to make the reflection functions (or moments or bimoments) corresponding to two "similar" atmospheres *about* equal, not strictly equal. This permits a certain latitude in the answer, so that more than one set of similarity relations may be put forward. Only reasons of convenience and numerical tests can establish a certain preference.

In an unbounded homogeneous medium with non-conservative scattering the equation of transfer permits a solution of the form

$$I(u, \tau) = P(u) e^{-k\tau} \quad (4)$$

where k , the diffusion exponent, is a positive constant and $P(u)$ is a positive function of the cosine u of the angle with the direction of increasing optical depth τ . The same solution with the signs of k and u inverted also exists. For some phase functions other solutions with larger k -values exist, in which, however, the corresponding $P(u)$ is partly negative. All of this is well-known theory (e.g. Van de Hulst, 1970). The k is an eigenvalue and $P(u)$ the corresponding eigenfunction of an integral equation. In the method of singular eigenfunctions, or Case method, all solutions must be found and used (e.g. Case and Zweifel, 1967, Kušćer and McCormick, 1974). In the present context only the least-damped one, defined above, is important.

The first similarity relation expresses the requirement that the exponential loss factor through the entire atmosphere should be the same:

$$b'k' = bk. \quad (5)$$

It has significance only for finite layers and may be used to find the new thickness b' , once the problem of finding a' and g' is settled. In order to solve that problem we shall use three integrals over $P(u)$:

$$1/2 \int_{-1}^1 P(u) du = 1 \quad \text{normalization integral,} \quad (6)$$

$$1/2 \int_{-1}^1 P(u) u du = \frac{1-a}{k} = y, \quad (7)$$

$$1/2 \int_{-1}^1 P(u) u^2 du = \frac{(1-a)(1-ag)}{k^2} \quad (8)$$

= D = "diffusion coefficient".

These equations are exact and follow by elementary means from the integral equation defining k and $P(u)$. Compare also Pomraning (1969).

The naming of the quantities thus defined requires some comment. The diffusion coefficient is a classical concept. It is the exact ratio between the "K-integral" and the "average intensity" in an unbounded medium. Its values for isotropic scattering have already been tabulated by Case *et al.* (1953). It is 1/3 for $a = 1$ and the systematic use of this value leads to the Eddington approximation.

When the similarity problem for non-conservative atmospheres was first posed (Van de Hulst and Grossman, 1968) it seemed obvious to define the new albedo a' by requiring that

$$y = y' \quad (\text{briefly: } y \text{ is invariant}). \quad (9)$$

For that reason the name similarity parameter was used for y . Invariant y means that the form of the diffusion pattern $P(u)$ remains roughly the same. But

other choices can serve the same purpose. The next simply choice would be to require that

$$\frac{D}{y} = \frac{1-ag}{k} \text{ is invariant.} \quad (10)$$

There is no obvious reason why one choice should be better than the other. Both are inconvenient in that they require the tabulation of the transcendental function $k(a, g)$ before the numerical value of a' (for given a, g , and $g' = 0$) can be determined. This problem bothered us for some time and numerical tests generally gave the impression that the two choices tended to give deviations in opposite directions so that a still better choice might be somewhere between.

A somewhat logical way out was found by considering the series expansions in terms of k , which are rapidly convergent whenever a is close to 1, i.e., for nearly conservative media. The known expansion of a (Van de Hulst, 1968) yields the leading terms:

$$y = \frac{k}{3(1-g)}, \quad D = \frac{1}{3}, \quad \frac{D}{y} = \frac{1-g}{k}.$$

The next terms, in each case, are of the order of k^2 times the leading terms. Using these expansions, we may now rewrite the choices made above in such a form that the quantities kept invariant have the same leading term, i.e. coincide in the limit $a \rightarrow 1$. This leads to:

$$\text{first choice (9): keep } y\sqrt{3} = \frac{(1-a)\sqrt{3}}{k} \text{ invariant,} \quad (11)$$

$$\text{second choice (10): keep } y/D\sqrt{3} = \frac{k}{(1-ag)\sqrt{3}} \quad (12)$$

invariant.

These expressions immediately suggest a third, intermediate choice, namely the geometric average of these quantities:

$$\text{third choice: keep } \frac{y'}{\sqrt{D}} = \sqrt{\frac{(1-a)}{(1-ag)}} = s \text{ invariant.} \quad (13)$$

This third choice has several attractive properties. It is intermediate and likely to be at least as good as the two others. Also, it can be used without first calculating a table of $k(a, g)$. Therefore, we propose that s may now be called the similarity parameter. The supreme test must however be made by means of numerical comparisons and we shall presently do so with the spherical albedo.

The practical determination of a' and b' (with adopted $g' = 0$) from a given set a, b, g , now proceeds as follows.

First find a' from the invariance of s , i.e.,

$$1-a' = \frac{1-a}{1-ag}, \quad a' = a \frac{1-g}{1-ag} \quad (14)$$

then find $k(a, g)$ and $k'(a', 0)$ exactly, and finally find b' from Eq. (5).

Optionally, the middle step may be replaced by the approximation

$$k(a, g) = \{3(1-a)(1-ag)\}^{1/2} \quad (15)$$

which is obtained from Eq. (8) by using for D the approximate value $1/3$, which is strictly valid for $a \rightarrow 1$. This approximation makes the invariant quantities in (11), (12) and (13) coincident. It further makes $bk = b(1-ag)s\sqrt{3}$ so that Eq. (5) is reduced to

$$b(1-ag) \text{ is invariant.} \quad (16)$$

In the limit $a \rightarrow 1$ we simply have $a' = 1$ and

$$b(1-g) \text{ is invariant.} \quad (17)$$

This quantity has also been called the "effective optical thickness".

3. A Universal Formula for the Spherical Albedo of a Homogeneous Semi-infinite Atmosphere

Values of A^* as well as many other quantities, accurate to 5 decimals, are available from a book in preparation (Van de Hulst, 1975) for the set of Henyey-Greenstein phase functions with $a = 0.99, 0.95, 0.9, 0.8, 0.6, 0.4, 0.2$ and $g = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}$. These computations were performed by Dr. K. Grossman. A 4-decimal table for $g = 0$, isotropic scattering, has been published by Chamberlain and Smith (1970). Incidental values are available for other phase functions and other values of a . This gives sufficient material for a rather thorough test.

Figure 1a shows the straight plot of A^* against $\sqrt{1-a}$. The variation with g is seen to be enormous. Figure 1b shows the same values plotted against the similarity parameter s defined by Eq. (13). The differences from one smooth curve are so small that on the scale of the graph they are invisible. If anything better than a 1% error would satisfy us, we might stop right here. However, we were curious to know the size of the actual deviations. This necessitated an interpolation formula, which was found as follows. First, there must be a factor $1-s$ to make sure that A^* vanishes at $a=0, s=1$. Secondly, the expansion of A^* (Van de Hulst, 1968; there A^* is called URU) in powers of k for an arbitrary phase function may be transcribed into an expansion in s of the form

$$A^* = 1' - (4/\sqrt{3})s + 4q's^2 + \dots \\ = 1 - 2.309s + 2.842s^2 + \dots \quad (18)$$

The coefficient of s is completely independent of the phase function and the coefficient of s^2 very nearly so. For the reduced extrapolation length $q' = q(1-g)$ stays well within the interval 0.71 ... 0.72 for a wide variety of phase functions (Van de Hulst, 1971). In

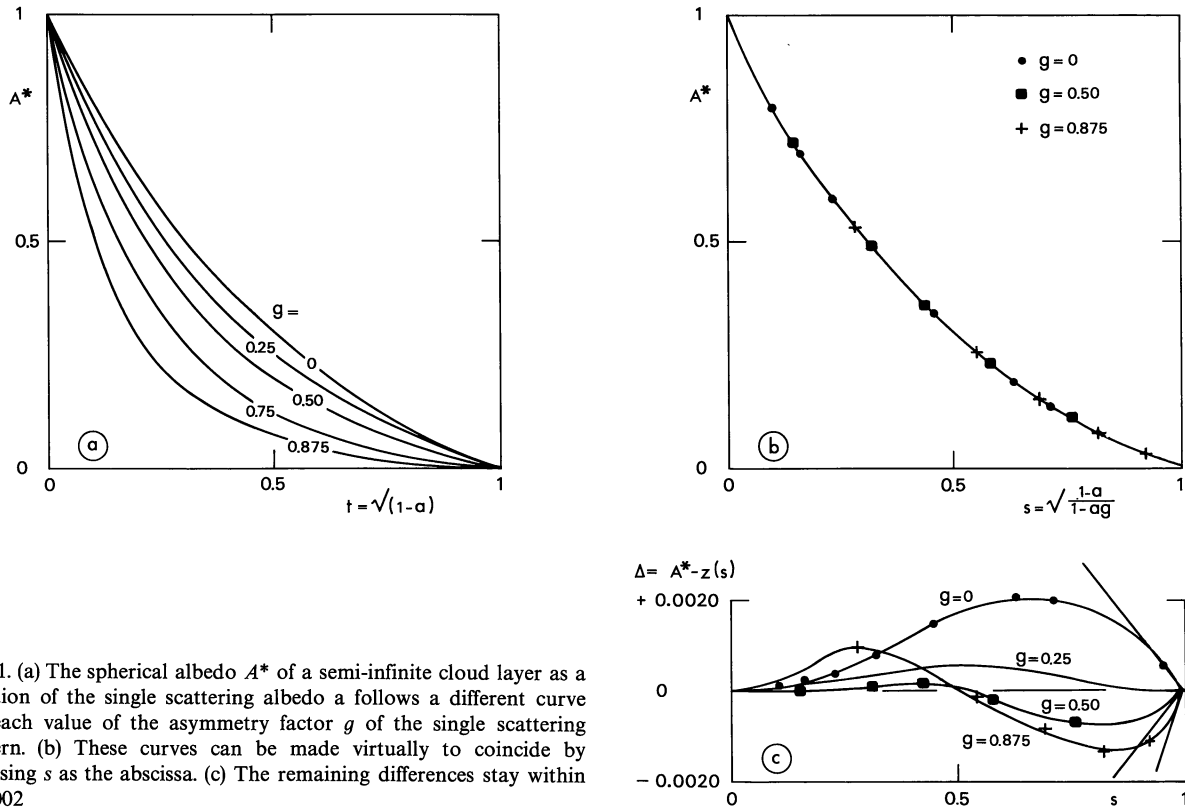


Fig. 1. (a) The spherical albedo A^* of a semi-infinite cloud layer as a function of the single scattering albedo a follows a different curve for each value of the asymmetry factor g of the single scattering pattern. (b) These curves can be made virtually to coincide by choosing s as the abscissa. (c) The remaining differences stay within ± 0.002

Eq. (18) we have used the q' for isotropic scattering. By postulating an equation of the form $(1 + c_1 s)(1 - s)/(1 + c_2 s)$ we can now find the values of c_1 and c_2 to match the two coefficients in Eq. (18). We rounded off and finally adopted

$$A^* \approx z(s), \text{ with } z(s) = \frac{(1 - 0.139s)(1 - s)}{1 + 1.17s}. \quad (19)$$

The numerical calculation of $z(s)$ gave in Fig. 1b a curve, which is again indistinguishable from the points. However, the differences $\Delta = A^* - z(s)$ can now be computed to 5 decimal places and are plotted in Fig. 1c on a scale 100 times enlarged with respect to 1b. The maximum error is seen to be 0.002. For $g = 0.875$ the number of available points is too small to draw a precise curve. Small computing errors are not excluded.

We examined also the alternative of plotting A^* against $u = [(1 - a)/(1 - g)]^{1/2}$ instead of s . The expansion in terms of powers of u , analogous to Eq. (18), has the same coefficients as (18) in the terms proportional to u and u^2 (Sobolev, 1969, 1974) but the coefficient of u^3 (Yanovitsky, 1972) is different from that of s^3 in (18). The plots against u (not reproduced) do not coincide very well. For instance, A^* near $u = 0.5$ ranges from about 0.30 ($g = 0$) to 0.34 ($g = 0.875$), nor do the curves reach a common end point at $a = 0$.

Table 1 provides some further comparisons. Wang (1972) derived a formula of A^* for which he claimed an accuracy better than one per cent over the entire range. Table 1, with more accurate "exact" values than those of Sobolev quoted by Wang, shows that this claim is correct but that the simpler function $z(s)$ comes much closer; it rarely needs corrections exceeding 0.001.

Table 1. Comparison with exact values. Δ = correction to be added to approximate formula to obtain exact value

Phase function	g	a	A^* exact	s	$z(s)$	$10^5 \Delta$	Wang Eq. (14)	$10^5 \Delta$
Isotropic	0	.975	.69500	.15811	.69485	+ 15	.692	+ 300
Isotropic		.9	.47802	.31623	.47716	+ 86	.472	+ 602
Isotropic		.5	.14652	.70711	.14453	+ 199	.139	+ 752
Isotropic		.1	.02170	.94868	.02111	+ 59	.020	+ 170
Rayleigh phase function	0	.975	.69	.15811	.69485	OK		
		.95	.60	.22361	.59626	OK		
Henyey-Greenst.	1/3	.9	.40897	.37796	.40865	+ 32		
linearly anisotropic	1/3	.9	.40834	.37796	.40865	- 31	.399	+ 934

The same table shows two entries for the Rayleigh-phase function with the "exact" values in 2 decimals taken from Table 25 of Harris (1961). The agreement is again perfect but the accuracy is not high enough to say anything more about Δ . In principle, it is possible to make a better comparison, because the reflection function for non-conservative scattering with an arbitrary linear combination of the Legendre functions or orders 0, 1 and 2 has been derived by Horak and Chandrasekhar (1961). We have integrated this expression according to Eqs. (2) and (3) and find

$$r(\mu) = 1 - (p + q\mu)H(\mu), \quad (20)$$

$$A^* = 1 - 2p\alpha_1 - 2q\alpha_2. \quad (21)$$

The righthand members are in Horak's and Chandrasekhar's notation. Unfortunately, no numerical computations for the non-conservative case seem to have been published.

The reflection against a semi-infinite atmosphere with non-conservative Rayleigh scattering, including polarization, has been solved completely with extensive and precise tables (Abhyankar and Fymat, 1970, 1971). But the integrations necessary to find the spherical albedo A^* have not been performed.

An accurate comparison for $g = 1/3$, also shown in Table 1, gives the expected result. The exact values for the two phase functions cited give corrections Δ of opposite sign but both only a fraction of one per cent. The value for the Henyey-Greenstein function is again based on computations by K. Grossman. The value for linearly anisotropic scattering was newly computed on the basis of familiar formulae (Chandrasekhar, 1950), using again Eq. (21), now with $q = 0$.

In the limit near $a = 0$, $s = 1$, single scattering forms the dominant term, resulting in the following linear approximations

$$z(s) = 0.3968(1-s), \quad 1-s = \frac{1}{2}(1-g)a,$$

$$a = 2(1-g)^{-1}(1-s)$$

H.G. phase functions

$$g = 0 \text{ (isotropic): } A^* = 0.2046 \quad a = 0.4092(1-s), \\ \Delta = +.0124(1-s),$$

$$g = \frac{1}{4}: \quad A^* = 0.1486 \quad a = 0.3963(1-s), \\ \Delta = -.0005(1-s),$$

$$g = \frac{1}{2}: \quad A^* = 0.0963 \quad a = 0.3852(1-s), \\ \Delta = -.0116(1-s),$$

$$g = \frac{3}{4}: \quad A^* = 0.04695 \quad a = 0.3756(1-s), \\ \Delta = -.0212(1-s),$$

$$g = \frac{7}{8}: \quad A^* = 0.02320 \quad a = 0.3712(1-s), \\ \Delta = -.0256(1-s).$$

These tangents have been drawn in Fig. 1c. A practical situation, in which Eq. (19) may be used in the reverse sense, is posed by the planet Venus.

Here A^* is measured, an estimate of g is known from other considerations, and the value of a is sought. Hansen and Hovenier (1974) solve this problem by getting from A^* first the value of a' correct for isotropic scattering and then $a = 1 - (1 - a')(1 - g)$, which is an approximation to Eq. (14) valid if a is close enough to 1. The biggest deviation could be expected in the ultraviolet, where starting from $A^* = 0.55$, $g = 0.761$, Hansen and Hovenier adopt $a = 0.98427$, whereas Eq. (19) gives $a = 0.98329$. This differences does not affect their conclusions.

4. The Plane Albedo

Naturally, the question arises whether a universal formula such as Eq. (19) for A^* , can also be derived for the plane albedo $r(\mu)$. Such a formula, if it exists, should have the form of a function of μ and s . Replacement of s by some other combination of a and g would make it impossible to find upon integration over μ the proper form for A^* . We therefore compared in Table 2, example A, the precise $r(\mu)$ values for two combinations of a and g that happen to have precisely the same value $s = (8/17)^{\frac{2}{3}}$. The spherical albedo, which is nearly 16 per cent, shows the same good match which we have already noted. The plane albedo does not match very well, which is not a great surprise since this example is very far from the conservative case. It is quite clear, for instance, that the plane albedo for grazing incidence, $r(0)$, depends chiefly on a and only weakly on g .

The only remedy is to add another independent variable, i.e., to find a convenient approximation formula of the form $r(\mu, s, a)$ or $r(\mu, s, g)$. This should be a relatively easy task, for it means *no* reduction of the number of variables from the original function so that, in fact, the similarity concept is dropped altogether. Closer similarity is found if we come closer to the conservative case, as shown by the example B of Table 2. This example is well enough in the nearly-conservative domain to approximate Eq. (14) by the rule that $(1-a)/(1-g)$ should be the same.

Table 2. Two examples by which the similarity for the plane albedo $r(\mu)$ may be tested

	Example A		Example B	
	$a = 0.6$ $g = 0.25$ $s = 0.6860$	$a = 0.9$ $g = 0.875$ $s = 0.6860$	$a = 0.96$ $g = 0$ $s = 0.2000$	$a = 0.99$ $g = 0.75$ $s = 0.1970$
$\mu = 0$.3593	.6255	.8000	.8624
0.1	.2932	.4269	.7598	.8033
0.3	.2229	.2658	.7043	.7285
0.5	.1781	.1836	.6593	.6710
0.7	.1462	.1331	.6207	.6222
0.9	.1223	.0993	.5868	.5791
1	.1125	.0865	.5713	.5592
A^*	.1580	.1568	.6306	.6348

This makes a difference of only $1\frac{1}{2}\%$ in s . The differences between the corresponding $r(\mu)$ values found in this example are 5% at most.

Altogether, we feel that an approximation formula with s as one of the independent variables should have a practical advantage. How to find such a formula is a matter of taste and may depend on the problem at hand. Starting by an approximation concept in solving the equation of transfer, as Wang has done, is certainly possible. Getting first the precise numerical result and fitting an expression by making incidental use of known limiting cases [the method followed above to find Eq. (19)] is also attractive and gives a better guarantee of a close match.

References

- Abhyankar, K.D., Fymat, A.L. 1970, *Astron. & Astrophys.* **4**, 101
 Abhyankar, K.D., Fymat, A.L. 1971, *Astrophys. J. Suppl.* **23**, 35
 Ambartsumyan, V.A. 1958, "Theoretical Astrophysics", London, Pergamon
 Case, K.M., De Hofman, F., Placzek, G. 1953, Introduction to the theory of neutron diffusion I, Los Alamos Sci. Lab.
 Case, K.M., Zweifel, P.F. 1967, Linear transport theory, Addison-Wesley, Reading
 Chamberlain, J.W., Smith, G.R. 1970, *Astrophys. J.* **160**, 755
 Chandrasekhar, S. 1950, Radiative transfer, Oxford, Clarendon Press
 Hansen, J.E., Hovenier, J.W. 1974, *J. Atmosph. Sci.* **31**, 1137
 Harris, D.L. 1961, in *The solar system*, Vol. III, Planets and satellites, G. P. Kuiper and B. M. Middlehurst, Ed., Univ. of Chicago, p. 272
 Horak, H.G. 1950, *Astrophys. J.* **112**, 445
 Horak, H.G., Chandrasekhar, S. 1961, *Astrophys. J.* **134**, 45
 Kušćer, I., McCormick, N.J. 1974 in The UCLA international conference on radiation and remote probing of the atmosphere, J. G. Kuriyan, Ed., Dept. of Meteorology Univ. California, Los Angeles, p. 196
 Pomraning, G.C. 1969, *Astron. & Astrophys.* **2**, 419
 Russell, H.N. 1916, *Astrophys. J.* **43**, 173
 Sobolev, V.V. 1969, *Doklady A.N.* **184**, 318, *Sov. Phys. Dok.* **14**, 1
 Sobolev, V.V. 1974, Light scattering in planetary atmospheres, W. M. Irvine, translator.
 Van de Hulst, H.C. 1968, *Bull. Astron. Inst. Neth.* **2**, 419
 Van de Hulst, H.C. 1970, *Astron. & Astrophys.* **9**, 366
 Van de Hulst, H.C. 1971, in Planetary Atmospheres, I.A.U. Symp. No. 40, Ed., Sagan, Owen and Smith, Dordrecht, Reidel
 Van de Hulst, H.C. 1975, Multiple light scattering in plane atmospheres
 Van de Hulst, H.C., Grossman, K. 1968, p. 35 in *The atmospheres of Venus and Mars*, Ed., J. C. Brandt and M. B. McElroy, New York, Gordon and Breach
 Wang, L. 1972, *Astrophys. J.* **174**, 671
 Yanovitsky, G. 1972, *Astron. Zh.* **49**, 844, *Sov. Astron. A. J.* **16**, 687 (1973)
 H. C. van de Hulst
 Sterrewacht
 Leiden-2401
 The Netherlands