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Duality in 2 + 1D quantum elasticity: superconductivity and quantum nematic order

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Abstract

Superfluidity and superconductivity are traditionally understood in terms of an adiabatic continuation from the Bose-gas limit. Here we demonstrate that at least in a 2 + 1D Bose system, superfluidity can arise in a strict quantum field-theoretic setting. Taking the theory of quantum elasticity (describing phonons) as a literal quantum field theory with a bosonic statistic, superfluidity and superconductivity (in the EM charged case) emerge automatically when the shear rigidity of the elastic state is destroyed by the proliferation of topological defects (quantum dislocations). Off-diagonal long range order in terms of the field operators of the constituent particles is not required. This is one of the outcomes of the broader pursuit presented in this paper. In essence, it amounts to the generalization of the well known theory of crystal melting in two dimensions by Nelson et al. [Phys. Rev. B 19 (1979) 2457; Phys. Rev. B 19 (1979) 1855], to the dynamical theory of bosonic states exhibiting quantum liquid-crystalline orders in 2 + 1 dimensions. We strongly rest on the field-theoretic formalism developed by Kleinert [Gauge fields in Condensed Matter, vol. II: Stresses and Defects, Differential Geometry, Crystal Defects, World Scientific, Singapore, 1989] for classical melting in 3D. Within this framework, the disordered states correspond to Bose condensates of the topological excitations, coupled to gauge fields describing the capacity of the elastic medium to propagate stresses. Our focus is primarily on the nematic states, corresponding with condensates of dislocations, under the topological condition that disclinations remain massive. The dislocations carry Burgers vectors as topological charges. Conventional nematic order, i.e., the breaking of space-rotations, corresponds in this field-theoretic duality framework with an ordering of the

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Burgers vectors. However, we also demonstrate that the Burgers vectors can quantum disorder despite the massive character of the disclinations. We identify the physical nature of the 'Coulomb nematic' suggested by Lammert et al. [Phys. Rev. Lett. 70 (1993) 1650; Phys. Rev. E 52 (1995) 1778] on gauge-theoretical grounds. The 2 + 1D quantum liquid crystals differ in fundamental regards from their 3D classical counterparts due to the presence of a dynamical constraint. This constraint is the glide principle, well known from metallurgy, which states that dislocations can only propagate in the direction of their Burgers vector. In the present framework this principle plays a central role. This constraint is necessary to decouple compression rigidity from the dislocation condensate. The shear rigidity is not protected, and as a result the shear modes acquire a Higgs mass in the dual condensate. This is the way the dictum that translational symmetry breaking goes hand in hand with shear rigidity emerges in the field theory. However, because of the glide principle compression stays massless, and the fluids are characterized by an isolated massless compression mode and are therefore superfluids. Glide also causes the shear Higgs mass to vanish at orientations perpendicular to the director in the ordered nematic, and the resulting state can be viewed as a quantum smectic of a novel kind. Our most spectacular result is a new hydrodynamical way of understanding the conventional electromagnetic Meissner state (superconducting state). Generalizing to the electromagnetically charged elastic medium ('Wigner Crystal') we find that the Higgs mass of the shear gauge fields, becoming finite in the nematic quantum fluids, automatically causes a Higgs mass in the electromagnetic sector by a novel mechanism.

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1. Introduction

The concept of duality has been around for a long time in high energy- and statistical physics, but it seems that the condensed matter community is still in the process of getting used to its powers. As compared to its use in fundamental field- and string theories, the concept acquires a quite vivid and physical interpretation in the condensed matter arena. Quantum field theories become of relevance as a description of the highly collective, long wavelength properties of systems composed of strongly interacting microscopic entities. Their emergence is tied to order. In their most straightforward interpretation, field theories enumerate the physics associated with spontaneous symmetry breaking. The fields are describing the Goldstone modes. Implemented in this context, duality acquires the meaning that states which appear as disordered are in fact also governed by order, although their order parameters might be beyond the reach of the most brilliant experimentalists. In continuum field theories this notion acquires a very precise meaning. Starting from an ordered state, disorder is governed exclusively by the singular (in fact, multivalued) configurations of the order field. These translate in 'disorder operators' carrying quantized topological charges and the only admissible states of disorder correspond with condensates of the topological excitations. Having identified the disorder fields, it is easy to deduce the physics of the disordered states using the weaponry of order physics.

Duality is especially powerful in 2 + 1 dimensions. The reason is that the topological excitations associated with conventional orders are typically point particles in 2 + 1 dimensions. The disorder field theories describing the collective properties of the 'disorder matter' are of the well understood Ginzburg–Landau–Wilson variety: they are conventional Bose condensates albeit in terms of topological matter. A popular example is vortex duality, describing in the quantum context [4,5] the disordering of a superconductor in a Bose Mott-insulator driven by the disordering influence of the charging energies on the phase order [6–10]. The disorder operators are the well known vortices carrying quantized magnetic flux, corresponding with particle-like excitations in 2+1D. At a critical charging energy these 'vortex bosons' proliferate in the vacuum and it can be shown (e.g., Appendix A) that the field theory describing this vortex matter is nothing else than the theory of a neutral superfluid, with a massless Goldstone boson describing the free propagation of the electromagnetic photon in the insulating state.

The most obvious form of order is, of course, crystalline order—the breaking of the spatial Euclidean group down to a lattice group. The first instance of a field theory describing a Goldstone sector is of course the theory of elasticity which emerged in the 19th century [11]. It is also the birthplace of topology in physics: the first topological defects which were appreciated as highly relevant are the dislocations and disclinations with their Burgers and Franck vectors as topological charges [12]. In the context of 2D classical phase transitions it formed the initial inspiration for the work of Kosterlitz–Thouless [13,14] placing the general duality motive on center stage in statistical physics. Subsequently the duality structure associated with the thermal melting of 2D solids was uncovered by Nelson et al. [1] (the KTNHY theory of 2D melting). In the 1980s the theory for 3D crystal melting was further developed, especially by Kleinert [15]. This strongly rests on the similarities with vortex duality in 3D, although it involves some non-trivial generalizations. The most complete treatment of this subject is found in the two volume textbook on gauge theories in condensed matter physics by Kleinert [2,7], where the second volume is dedicated to elasticity.

Dynamics is not the same as statics and it is remarkable that the elastic analogue of the quantum-mechanical vortex duality in 2 + 1D goes largely unexplored [16]. As the KTNHY theory already makes clear, liquid crystalline orders appear in a natural way within the duality framework. As compared to the standard lore in terms of rod-like molecules aligning their long axis, etc., a considerable shift of interpretation occurs. Nematic (and as we will see, also smectic) orders appear as a consequence of the rich topological structure associated with spatial symmetries. These can be entirely classified in topological terms. Nematics are those states of matter characterized by a condensation of dislocations while disclinations are massive excitations: the 'hexatic' state predicted by KTNHY. The isotropic fluid is a state where both dislocations and disclinations are condensed and because these are interdependent, it is best termed a 'defect condensate', see [2].

Our interest was originally triggered by several suggestions regarding the possible existence of zero temperature quantum manifestations of liquid crystalline order. These appeared at more or less the same time, although aimed at different physical circumstances. Mullen et al. [17] suggested a possible super-hexatic phase in 2D Helium [18]. Balents and Nelson [19] investigated a smectic phase in flux systems and its quantum analogue. Kivelson et al. [20] advanced the possible existence of smectic and nematic orders in cuprate superconductors, shortly thereafter followed by suggestions regarding smectic and nematic quantum Hall stripe phases [21] (see also [22,23]). Viewed from a broader perspective, these ideas are part of the current development in condensed matter physics to search for quantum fluids which are correlated to a degree that the conventional quantum gas perspective (Fermi-liquids, the Bogoliubov Bose-gas, BCS theory) is no longer relevant. Instead, it might be more appropriate to view such systems as on the verge of becoming ordered ('fluctuating order', in the context of high T_c superconductivity see [24,25]). In a broader sense, one might want to equate this 'fluctuating order' to duality, as all that exists is order and the disorder derivatives of order. All degrees of freedom are of a collective kind and the system at long distances has completely forgotten about the microscopic constituents (like the electrons, Cooper pairs).

As we will discuss in detail, 'fluctuating order' acquires a very precise meaning in the context of the quantum melting of a crystal. We will in this paper develop the theory of quantum liquids which are in a literal way derivatives from the collective fields of the solid: the Goldstone modes (phonons), and the topological defects (dislocations, disclinations). In a precise way, our construct exaggerates the orderly nature of matter to an extent that the limiting case we describe cannot be realized literally in any condensed matter system. The degrees of freedom of the constituent particles (the 'interstitials'; the cooper pairs, He atoms, etc.) are not included in this theory. However, the case can be made that the degrees of freedom of the interstitials are liberated at the moment that the solid undergoes quantum melting. These degrees of freedom are of relevance for the long wavelength physics, while they are not an intrinsic part of the field theoretic description. This is a main short coming of our approach and the reader should view it as an exposition of an unphysical limit which can nevertheless be closely approached, at least in principle. It should also be regarded as complementary to much of the existing work on quantum liquid crystals which has a (implicit) focus on the physics associated with the interstitials [17,19,20,26,27]. This also includes D.-H. Lee's stripe-superconductivity duality [28] which is also of the interstitial kind, and obviously the rich body of work explicitly dealing with supersolid physics [29–32].

Besides the neglect of interstitial excitations, there are a number of other limitations to this work. Most importantly, we limit ourselves to bosonic matter, for the usual reason that we do not know how to deal with the fermionic minus signs (see [33] for some intriguing results on fermionic liquid crystals). The other limitations are less essential and are motivated by technicalities. We limit ourselves to $2+1$ dimensions because dislocations and disclinations are particle-like in this dimension, and the general form of the disorder field theory is fully understood. In $3+1$ dimensions the defects are strings (dislocation, disclination loops in 3D space). Although there is every reason to expect that the outcomes will be qualitatively the same, it is not known how to construct string condensates explicitly because of the lack of a second quantized formalism. Next, our focus will be entirely on the nematic states, i.e., the dislocation condensates, for no other reason than to keep the scope of this paper limited. We do not address therefore in any detail the disorder condensates associated with the proliferation of disclinations. In fact, because rotations are Abelian in two spatial dimensions, not much interesting is expected to happen when disclinations proliferate, as associated with the melting of the quantum nematics into the isotropic state. This is different in three spatial dimensions where the non-abelian nature of the rotational group comes into play leading to interesting braiding and multiplicity of the disclinations structures. Mathematical methods in the form of quantum double symmetries recently emerged allowing a possible systematic investigation of such topological structures [34], and we defer this to a future study. Finally, we limit ourselves to the theory of the isotropic quantum-elastic medium (i.e., showing global rotational invariance, 'no crystal faces') and this is merely for convenience. Our findings are easily generalized to any 2D space group.

Within these limitations, there is much to be discovered. In technical regards, we rest strongly on the mathematical methodology developed by Kleinert [2]. Our work can be viewed as an application of his methods in a somewhat altered context, and we will review this methodology in some detail in Section 2 to make this paper self-contained. At the core of this methodology is the realization that the universe of dislocations and disclinations is governed by gauge fields. The defects act like sources in electromagnetism, exerting long range forces on each other by the exchange of 'photons.' It is, in itself, an entertaining exercise to rewrite the familiar physics of dynamical phonons in the language of 'stress photons' residing in $2+1$D space–time as we discuss in Sections 4 and 5. These stress photons become quite meaningful when the solid quantum-melts due to the proliferation- and Bose condensation of the dislocations. The dislocations carry shear 'stress' charge and the effect is that together with the stress-gauge fields the analogue of an electromagnetically charged superconductor emerges. This exhibits a Meissner effect (Higgs phenomenon) causing the shear stress photons to become massive. This represents the physical fact that in the fluid the shear rigidity associated with the elastic state becomes short ranged.

The informed reader will recognize the similarity with vortex duality. However, in two regards the dislocation condensate is radically different and more interesting than the vortex condensate. First, vortices carry scalar charges while dislocations carry vectorial charges, the Burgers vectors. Under the condition that the disclinations are massive, this vectorial nature of the charge does not pose a difficulty because the governing symmetry stays Abelian (pure translations). The Burgers vectors can be viewed as additional degrees of freedom, responsible for the various forms of nematic order. In Section 3 we will develop the basic formalism underlying the duality. We find that besides the conventional form of nematic order (breaking spatial rotational symmetry, the $2+1$D generalization of the hexatic of 2D), a purely topological form of nematic order is to be expected: although rotational symmetry is unbroken, disclinations are still massive. This 'Coulomb nematic' was predicted on different grounds some time ago by Lammert et al. [3] and we find a natural physical interpretation for it in our duality framework.

The other novelty refers to a very special phenomenon only occurring on the dynamical level, involving the time axis. On the quantum level this becomes most central, because space and time are entangled: it is the 'glide' principle, referring to the fact well known to metallurgists [12] that dislocations can only propagate in the direction of their Burgers vector. This glide constraint plays a central role in the remainder and it renders our theory to be a real quantum theory, having no classical analogy.

Glide has a remarkably deep meaning. We will prove that the glide constraint is a necessary condition for the decoupling of compressional stress from the dislocation currents. This has far reaching consequences. It is a textbook wisdom that translational symmetry breaking is associated with long range shear rigidity. Upon melting the solid, the medium loses its capacity to transmit shear forces. However, liquids still carry sound, meaning that they are characterized by a massless compression mode. On the quantum level, glide is needed to protect this compression mode against the dislocation condensate!

A first direct consequence of glide is that the nematic breaking rotational symmetry turns into a state which in a special sense is more like a smectic, as will be discussed in Section 8. The director order correspond with a rotational order of the Burgers vectors, and these in turn direct the 'superfluid' dislocation currents into one particular direction. This has the implication that the shear Higgs gap vanishes in the direction exactly perpendicular to the director where the phonons of the solid re-emerge.

We perceive the other consequence of glide as of a great general importance. We will derive the excitation spectrum of the dislocation condensate explicitly and this will turn out to be of a quite universal form (Sections 6 and 7). Besides an overall scale vector (phonon velocity), it is completely determined by the Poisson ratio and a single length, the 'shear penetration depth.' It consists of two massive shear modes and, in addition, a massless mode which is purely compressional.

In textbooks on quantum fluids, the emphasis is on the concept of off-diagonal long range order (ODLRO) as introduced by Penrose to understand the fundamental nature of the superfluid state. However, this way of thinking rests in last instance on the continuation to the bose gas limit and the present field theory is explicitly constructed so that this adiabatic connection is disrupted. However, Landau [35] and Feynman [36] constructed an alternative, purely *hydrodynamical* description of superfluidity. As we will discuss in detail in Section 7.3, this rests on the assumption that the superfluid is characterized by an isolated, propagating compression mode ('phonon') in the scaling limit. This is consistent with the ODLRO in the Bose-gas. However, the present field theory shows that the Landau/Feynman hydrodynamic theory is in a sense incomplete. Without referral to microscopics it turns out to be possible to precisely specify in hydrodynamical terms where the propagating compression is coming from:

> it is sufficient condition for the existence of the superfluid state that a solid looses its rigidity to shear stresses due to condensation of dislocations.

This is just what is happening in the present field theory. The dislocation condensate is characterized by an isolated massless compression mode, and according to hydrodynamics it has to be a superfluid. However, it is a superfluid which is not based on 'conventional' ODLRO. The ramification is that the Landau/Feynman understanding of superfluidity is more general than the one based on ODLRO, because we have identified a superfluid which is disconnected from the Bose-gas limit!

A next issue is what happens when electromagnetism is coupled in, i.e., considering the duality for an electromagnetically charged ('Wigner') bosonic crystal. If the dual is indeed a superfluid, the charged case should exhibit an electromagnetic Meissner effect. General hydrodynamical arguments are available demonstrating that 'our' superfluid should turn into an electromagnetic Meissner–Higgs state. These are of a much more recent origin. In 1989 Wen and Zee [37] demonstrated that the presence of an isolated massless compression mode in the neutral case is sufficient condition for the appearance of a Meissner–Higgs state when electromagnetism is coupled in. This argument rests on duality: the compression mode can be dualized in a pair of compression stress photons (see Section 3.5) which are coupled to EM photons. Integrating the former yields a Higgs mass gap for the latter.

Although consistent with the Wen–Zee theorem, we find that in the full elastic duality a EM Higgs mass is generated by yet a different mechanism. We percieve this as of great general importance and this counterintuitive affair is presented in Section 10. By just coupling the electrical fields to the displacement fields of the Wigner crystal, we find that a miracle occurs upon dualizing to a stress-photon representation. Automatically, a Meissner term appears acting on the electromagnetic vector potentials. At first sight, it appears as if the crystal is already exhibiting a Meissner effect yet this is not quite the case. The electromagnetic photons are linearly coupled to the stress photons. Upon integrating these out, a counter electromagnetic Meissner term is generated which is exactly cancelling the 'bare' Meissner. However, for this compensation to work, the shear photon needs to be massless. When the shear-mass becomes finite because of the dislocation condensate this compensation is no longer complete and the system turns into a superconductor. In summary, in the stress-gauge field representation, the Meissner effect lies in hide in the solid to get liberated in the superconductor because shear rigidity becomes short range.

It appears that this mechanism is of a most general nature. The specifics of the nematic states etc. do not enter the arguments leading to these conclusions in any obvious way. In fact, we are under the impression that it offers a deep, hydrodynamical, insight into the nature of superconductivity, and the Higgs phenomenon in general. Giving matters a further thought, it appears that this mechanism is in not at all conflicting with the usual understanding in terms of off-diagonal long range order in terms of the constituent bosons. The latter is just a special case of the former, derived from the gas limit where the shear length is vanishingly small.

To further emphasize this issue, we ask under what circumstance can one see the difference between the conventional superconductor and our 'order superconductor.' The answer is that the superconductor is characterized by *two length scales*: the magnetic penetration depth and the shear penetration depth. In the conventional description the latter is just ignored. However, in our formalism the shear length takes a central role and we can study explicitly what happens when the shear-length exceeds the bare magnetic penetration depth. We find new physics (Section 9). In this regime the effective penetration depth turns into the geometrical mean of the shear- and bare magnetic lengths. More strikingly, by a novel mechanism, the magnetic propagator acquires poles with real parts leading to oscillating magnetic screening currents. Upon entering the superconductor one finds a pattern of screening and anti-screening currents.

Finally, we will conclude this paper with a discussion of potential ramifications of our work in both the condensed matter- and cosmological context (Section 10).

2. General considerations: elasticity as a genuine field theory

The theory of elasticity is, of course, overly well known. Here we will interpret it as a literal quantum field theory. In doing so, the theory acquires the status of a toy model which might be good enough to reveal some most general features but it cannot be applied literally to any circumstance encountered in nature. The theory describes the long wavelength collective behaviors of crystals: matter spontaneously breaks spatial rotations and translations down to a lattice group. Historically, it is the first 'emergent' field theory which was discovered. In a much more modern setting, it is fascinating that it can be reformulated in a differential geometric language, with the outcome that at least the 3D isotropic theory turns out to be Einstein–Cartan gravity in 2 + 1D, with the disclinations and dislocations taking the role of curvature- and torsion sources, respectively [2,38–40]. In part motivated by the present work, interesting connections with quantum gravity have been discussed in a recent paper by Kleinert and one of the authors [JZ] [41]. Except for a short discussion in the conclusion section, we will ignore these aspects and instead focus on 2 + 1D quantum elasticity. This theory remembers its origin in the physics of non-relativistic particles with the consequence that Lorentz invariance is badly broken: the worldlines of non-relativistic particles are incompressible along the time direction and this fact is remembered by the long wavelength theory.

2.1. Some basics and definitions

Let us first summarize some basics of elasticity theory [2,11], to remind the reader and to introduce notations. Starting with the crystal, the theory is derived by asserting that constituent particle n has a equilibrium position $\vec{x}_n$, and a real coordinate $\vec{x}'_n = \vec{x}_n + \vec{u}_n$ (Fig. 1). Strictly speaking $\vec{u}_n$ should be finite in order to have a meaningful continuum limit. At distances large compared to the lattice constant a, one can define a displacement field $\vec{u}(\vec{x},\tau)$ such that $\vec{x}'(\tau) = \vec{x}(\tau) + \vec{u}(\vec{x},\tau)$ (τ is imaginary time; $\vec{u} = (u^x, u^y)$; the Latin labels $\sim a, b, \ldots$ refer to space coordinates; Greek labels $\sim \mu, \nu, \ldots$ refer to space–time; Einstein summation conventions are used everywhere). The distance vector between two material points at $\vec{x}$ and $\vec{y}$ is changed from $\mathrm{d}\vec{x} = \vec{x} - \vec{y}$ to $\mathrm{d}x'_a = \mathrm{d}x_a + \partial_b u^a \mathrm{d}x_a$, and its length from $\mathrm{d}r = \sqrt{\mathrm{d}\vec{x}^2}$ to $\mathrm{d}r' = \sqrt{\mathrm{d}\vec{x}^2 + 2w_{ab}\mathrm{d}x_a\mathrm{d}x_b}$ where the strain tensor w_{ab} is in linear approximation,

$$w_{ab} = \frac{1}{2}\left(\partial_b u^a + \partial_a u^b\right). \tag{1}$$

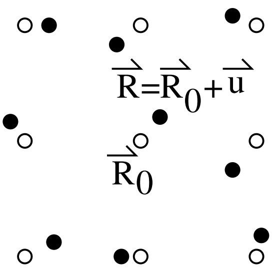

Fig. 1. In a crystal, the positions of 'atoms' (or electrons, Cooper pairs etc., the black dots) can be uniquely related to equilibrium positions forming a regular pattern (open circles) by displacements $\vec{u}$. A requirement for the crystal to exist is that the displacement field $\vec{u}(\vec{x}, t)$ is non-singular.

These strain fields are the physically meaningful entities entering the theory, because the elastic action can only depend on the differences $\mathrm{d}l - \mathrm{d}l'$. For fundamental geometrical reasons, this tensor is symmetric in the spatial indices. The part antisymmetric in the space indices corresponding with the rotational field, in 2 + 1D,

$$\omega_\tau = \frac{1}{2}\epsilon_{\tau ab}\partial_a u^b = \frac{1}{2}\left(\partial_x u^y - \partial_y u^x\right) \tag{2}$$

cannot enter the action as $(\omega_\tau)^2$ does not respect rotational invariance. Nevertheless, upon considering higher orders in the gradient expansion ('second gradient elasticity', see [2]) terms of the form

$$S \sim \int \mathrm{d}\Omega \xi_R^2 C_R (\partial_a \omega_\tau)^2 \tag{3}$$

will be encountered in the action. These invariants express the 'rotational stiffness' of the elastic medium. However, as compared to the leading order ('first gradient elasticity'), invariants $\sim w_{ab}^2$, these involve two extra gradients meaning that this rotational stiffness becomes only observable at length scales smaller than the length ξ_R. Hence, it is of no consequence to the long wavelength dynamics, but as we will see later, the finiteness of such terms is sufficient condition for the existence of nematic states.

For notational purposes, we will adopt the following convention. Asymmetric tensor fields are written as

$$F_\mu^a = \partial_\mu u^a, \tag{4}$$

such that, for instance, $\dot{u}^x = \partial_\tau u^x = w_\tau^x$, etc. The symmetrized fields are written as

$$F_{ab} = \frac{1}{2}(\partial_a u_b + \partial_b u_a) = \frac{1}{2}\left(w_a^b + w_b^a\right) \tag{5}$$

In the leading order gradient expansion, the (Euclidean) action should be $\sim w^2$. As no displacements in the time direction are allowed ($u^\tau = 0$) the kinetic energy density is simply $-\frac{\rho}{2}(\partial_\tau u^a)^2 = -\frac{\rho}{2}(w_\tau^a)^2$, while the potential energy density $C_{abcd} w_a^b w_c^d$ where the C's are the elastic moduli, and the theory of quantum elasticity becomes in Euclidean path integral form,

$$
\begin{aligned}
Z &= \int \mathcal{D}u^a \mathrm{e}^{-(1/\hbar)S_{\mathrm{elas}}}, \\
S_{\mathrm{elas}} &= \int \mathrm{d}\Omega \left[\frac{1}{2} C_{abcd} w_a^b w_c^d + \frac{\rho}{2} \left(w_\tau^a \right)^2 \right],
\end{aligned} \tag{6}
$$

where $\mathrm{d}\Omega = \mathrm{d}^2x\,\mathrm{d}\tau$ is the 2 + 1D Euclidean space–time volume element. In the remainder, we will set $\hbar = 1$ and consider most of the time just the form of the action S, leaving the path-integration (including the measures) implicit.

Let us now specialize to the theory of isotropic elasticity, describing the elastic medium which is invariant under global spatial rotations. This isotropic elasticity is the standard theory used by metallurgists for the reason that metals like steel are amorphous on macroscopic scales. For single crystals this is not a valid assumption, except for the 2D closed packed triangular crystal where, for accidental reasons, the gradient expansion yields the isotropic theory. We focus on this particular case, because it is the most basic and our findings are easily generalized to the less symmetric cases.

The isotropic medium can be parameterized in terms of two independent moduli: the compression- and shear moduli κ and μ, respectively. κ parameterizes the response associated with uniform compression, while shear (μ) is associated with the response arising from moving opposite sides of the medium in opposite directions. Compressional rigidity is universal in interacting non-relativistic matter. Shear is the rigidity exclusively associated with the breaking of translational invariance. The two respective compression and shear moduli are related to the Poisson ratio v via

$$
\kappa = \mu \frac{1+v}{1-v}, \tag{7}
$$

and the potential energy density can be compactly written as

$$
\mathcal{L}_{\mathrm{pot}} = \mu \left(\sum_{ab} w_{ab}^2 + \frac{v}{1-v} \left(\sum_a w_{aa} \right)^2 \right), \tag{8}
$$

using explicit summations for clarity. Combining this with the kinetic energy, we obtain

$$
\begin{aligned}
S = \mu \int \mathrm{d}\Omega \bigg[& \left(w_x^x\right)^2 + \left(w_x^y\right)^2 + \left(w_y^x\right)^2 + \left(w_y^y\right)^2 + \frac{1}{c_{\mathrm{ph}}^2} \left(\left(w_\tau^x\right)^2 + \left(w_\tau^y\right)^2 \right) \\
& + \frac{v}{1-v} \left(w_x^x + w_y^y \right)^2 \bigg],
\end{aligned} \tag{9}
$$

where the ‘phonon’ velocity

$$
c_{\mathrm{ph}}^2 = \frac{2\mu}{\rho}. \tag{10}
$$

In the remainder we set this velocity to one, unless stated otherwise. Time is measured in units of length, with c_{ph} being the conversion factor.

Fourier transforming to momentum-(Matsubara) frequency space $(\vec{q}, \omega)$, this action is diagonalized by a transversal-longitudinal (T, L) projection of the Fourier components u^a of the displacement fields,

$$\begin{aligned} u^x &= \hat{q}_x u^L + \hat{q}_y u^T, \\ u^y &= \hat{q}_y u^L - \hat{q}_x u^T, \end{aligned} \tag{11}$$

where $\vec{\hat{q}} = (\hat{q}_x, \hat{q}_y)$ is the unit vector in momentum space. The end result is

$$S = \mu \int \mathrm{d}^2 q \, \mathrm{d}\omega \left[\left(\frac{q^2}{2} + \omega^2 \right) |u^T|^2 + \left(\frac{q^2}{1-\nu} + \omega^2 \right) |u^L|^2 \right] \tag{12}$$

and one directly recognizes the transversal and longitudinal acoustic phonons propagating with velocities $c_T = c_{\mathrm{ph}}/\sqrt{2} = \sqrt{\mu/\rho}$ and $c_L = \sqrt{2\mu/(1-\nu)\rho}$, respectively.

Finally, for future use, let us consider the longitudinal and transversal strain propagators, associated with the dynamical form factor as measured in, e.g., neutron scattering. The propagators are defined as usual (see Appendix D) and the longitudinal (L) and transversal (T) propagators are,

$$\begin{aligned} G_L &= q^2 \langle\langle \hat{q}_x u^x + \hat{q}_y u^y | \hat{q}_x u^x + \hat{q}_y u^y \rangle\rangle, \\ G_T &= q^2 \langle\langle \hat{q}_y u^x - \hat{q}_x u^y | \hat{q}_y u^x - \hat{q}_x u^y \rangle\rangle, \\ G &= G_L + G_T, \end{aligned} \tag{13}$$

where G is the total propagator. It immediately follows that

$$\begin{aligned} G_L &= \frac{1}{\mu} \frac{q^2}{(q^2/(1-\nu)) + \omega^2}, \\ G_T &= \frac{1}{\mu} \frac{(q^2/2)}{(q^2/2) + \omega^2} \end{aligned} \tag{14}$$

describing the phonon poles.

2.2. The singularities

Thus far we reviewed the simple theory of phonons. Matters become more interesting considering how this medium can be destroyed. As long as the displacement field is finite, we can uniquely associate a particular particle to a particular site in the crystal such that translational invariance remains broken, and quantum elasticity remains asymptotically exact because of Goldstone protection. Hence, the question concerns the nature of the singularities of the displacement fields. These come in two classes, the non-topological interstitial excitations and the topological dislocations and disclinations:

(i) The non-topological defects—vacancies and interstitials. An individual particle can meander away from its lattice position, leaving behind a vacancy and moving via interstitial positions infinitely far away from its original location (Fig. 2). The displacement $\vec{u}_n$ of this particular particle becomes infinite, but the continuum field

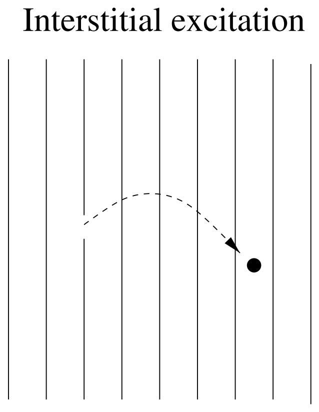

Fig. 2. Individual atoms can leave their equilibrium positions, leaving behind a vacancy-interstitial pair, called in this paper 'interstitial.' These point-like defects cannot be avoided at any finite temperature while at zero temperature their condensation causes the supersolid. As in other pictures, the lines just refer to 'rows of atoms.'

$\vec{u}(x)$ is by its very definition incapable of keeping track of this type of singularity. Hence, interstitials live 'outside' the realm of field theory, and separate degrees of freedom have to be introduced to keep track of their physics. Since interstitials and vacancies are point particles carrying a finite mass, they will always occur at a non-zero temperature. Therefore, a real crystal at room temperature is in fact a gas of interstitials coexisting with the ideal crystal. This coexistence is possible because the interstitials just 'dilute' the crystal, decreasing the amplitude of the order parameter, and only at a density of order unity a transition will follow to a fluid state. At zero temperature, matters are more sharply defined. At small coupling constant, interstitial-vacancy pairs will only occur as virtual excitations, forming closed loops in space–time, and the crystal has a precise definition. At a critical coupling constant these loops will blow out and interstitials will proliferate in the vacuum. The system turns into a superposition of a crystal and a quantum gas of interstitials. If these are bosons, the gas of interstitials will Bose condense and a supersolid will form. Finally, a transition follows where the amplitude of the crystal order parameter tends to zero and an isotropic superfluid will form (Fig. 3).

Although quite popular in condensed matter physics (e.g. [4,28–32]), this 'supersolid' alley is not generic. In order for this to work, interstitials should have a small mass relative to their topological competitors, and this is in practice only possible in the presence of strong Umklapp scattering. Strong interactions between the bosons and an external potential are needed, and these ingredients are wired in popular toy models like the Bose–Hubbard model.

(ii) The topological excitations occurring in the elastic medium are the dislocations and disclinations, associated with the restoration of translational- and rotational invariance, respectively. These are both point-particle-like in $2+1$D suggesting that their associated disorder field theories are relatively simple—in fact

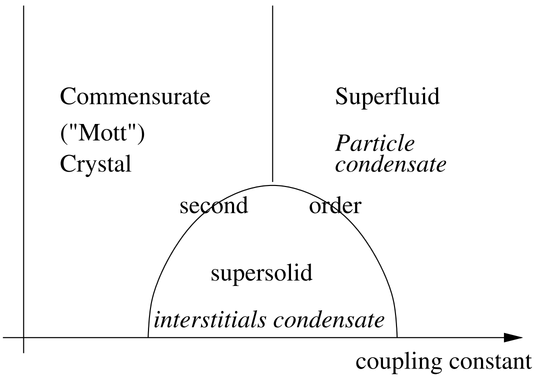

Fig. 3. The generic phase diagram of a system of bosons subjected to a lattice potential. When the kinetic energy is small (small coupling constant) the Bosons will localize forming a Mott-insulating state. Upon increasing the kinetic energy, a transition will follow into the superfluid state. However, under the right conditions (finite thermodynamic potential, finite range interactions) it might well happen that in between the Mott state and the superfluid a coexistence regime emerges: the supersolid, which can be viewed as a coexistence of a crystal and a Bose-gas of interstitials.

our main reason to specialize to this dimension. However, as discussed in great detail by Kleinert [2], the tensorial nature of the theory complicates matters greatly: for instance, the observation that the full disorder theory has to do with Einstein–Cartan gravity. Contrary to the interstitials, the topological singularities are a natural part of the full field theory, and they enumerate the singularities of the continuum displacement fields $\vec{u}$. The topological invariants in 2 + 1D of the dislocation (the Burger two component vector $\vec{n} = (n^x, n^y) = |n|(\hat{n}^x, \hat{n}^y)$, $\vec{\hat{n}}$ being the unit vector) and the disclination (the Franck 'scalar' Ω) can be defined through circuit integrals (Fig. 4),

$$\oint \mathrm{d}u^a = n^a, \qquad \oint \mathrm{d}\omega_R = \Omega, \tag{15}$$

where $\omega_R = \frac{1}{2}(\vec{\nabla} \times \vec{u})$, the rotation of $\vec{u}$ on the time slice (Eq. (2)). Using Stokes theorem, these can be written as (tensorial) dislocation- (J^a_μ) and disclination currents (I_μ) as

$$J^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda u^a, \qquad I_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\omega_R. \tag{16}$$

These topological currents are not independent (the proper current is the 'defect density', see Section 3.3). Amongst others, the disclination can be seen as a bound state of dislocations with parallel Burger vectors, while at the same time dislocations can be regarded as bound disclination- anti-disclination pairs. Disclinations act as sources for dislocations,

$$\partial_\mu J^a_\mu = -\varepsilon_{a\nu} I_\nu, \tag{17}$$

with $\varepsilon_{a\nu}$ the 2D anti-symmetric tensor.

Disclinations are the bad players, greatly complicating the theory. Two simple statements can be made directly. When dislocations and/or disclinations proliferate

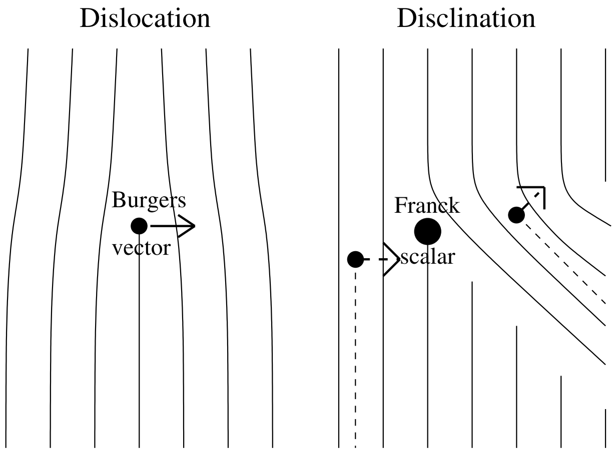

Fig. 4. As before, for convenience, the crystal lattice is encoded in lines referring to 'rows of atoms.' A dislocation corresponds with a half infinite row of atoms coming to and end somewhere in the solid. It is the topological defect associated with translations and its topological charge is clearly directional (the Burgers vector). The disclination is associated with a rotation and it can be viewed as a wedge of additional matter inserted by a Voltera process. Alternatively, it can be viewed as a bound state of dislocations with parallel Burgers vector. By inserting test dislocations one infers that their Burgers vector is transported as if a curvature source resides at the disclination core. The opening angle corresponds with its topological charge, the Franck scalar (in 2 + 1D).

in the vacuum, the crystal is immediately destroyed because of their topological nature. Further, when they proliferate together the phase transition will, under all circumstances, be first order. One might want to view conventional solid–liquid transitions in this way yet this notion is not very useful as the transition is strongly first order and the physics stays near the lattice constant, rendering field theory meaningless.

Our central assumption is that a regime exists where disclinations continue to be massive excitations, even when dislocations proliferate. If this is the case, it follows from Eq. (17) that the dislocation currents are conserved,

$$\partial_\mu J^a_\mu = 0. \tag{18}$$

This Bianchi identity simplifies the theory to such an extent that it becomes completely tractable. The theory describing the proliferation of dislocations becomes a straightforward extension of the well known Abelian–Higgs duality, reviewed in Appendix A. By definition, we call disordered states satisfying this topological condition nematic (quantum) fluids.

The 'first gradient' theory Eq.(9) governing the long wavelengths regime cannot supply the ingredients needed to distinguish the dislocation and disclination masses. The reason is discussed in Section 2.1. The lowest order theory does not allow an invariant expressing rigidity associated with rotations. According to Eq. (3), such a rigidity shows up first in second gradient elasticity and starts to exert its influences at length scales less than the length ξ_R. Since disclinations are rotational defects one anticipates that their energetics is influenced by the presence of such terms. As Kleinert [2] shows, this is indeed the case and he demonstrates that a sufficiently large rotational stiffness C_R is sufficient condition for the existence of a nematic regime where Eq. (18) is obeyed. This shows that nematic states are entities which can be addressed within continuum field theory. In condensed matter physics, however, the issue will be decided by the specifics of the physics at the lattice constant. It is questionable if the right conditions are ever met starting with a closed pack lattice of spherical particles interacting via van der Waals forces—the existence of a (super) hexatic state in 2 + 1D Helium is still subject of controversy [2]. On the other hand, as Kivelson et al. pointed out, this might be quite different starting out with the strongly anisotropic stripe-crystals as found in high T_c cuprates and quantum–Hall systems [20].

Let us assume that the right microscopic conditions are present. In addition, it appears to be necessary to explicitly forbid interstitials in order to keep full control of the theory. Under these circumstances, one expects a phase diagram with a topology as indicated in Fig. 5. When the rotational stiffness is to small, a first order transition from the crystal to the isotropic quantum fluid should occur when the coupling constant increases. In the field theory, this isotropic fluid corresponds with a combined dislocation/disclination ('defect') condensate. Upon increasing the rotational stiffness, a tricritical point will occur where the first order line bifurcates in two second order lines. The crystal first melts into a nematic fluid, corresponding with a dislocation condensate, and at a larger coupling constant a transition follows to the isotropic fluid. This is of course nothing else than the straightforward extension of the KTNHY theory of 2D classical melting [1] to the zero-temperature, 2 + 1D quantum melting context.

This phase diagram is not surprising, and it should be taken as an input for what follows. At this stage, it should come as a surprise to the reader that we call both the

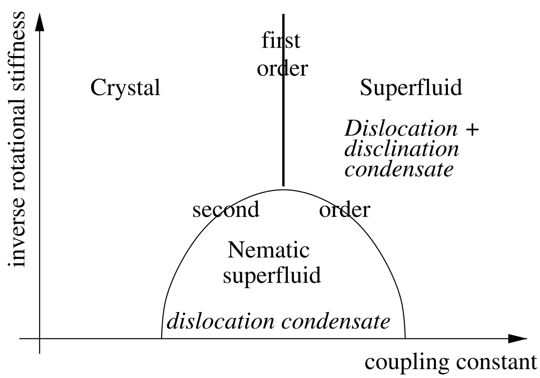

Fig. 5. The generic phase diagram associated with 2 + 1D Bosonic matter, assuming that the field theoretic description based on quantum elasticity is of relevance. When the rotational stiffness is small a first order transition from the crystal directly into the superfluid is expected, corresponding with a simultaneous Bose condensation of dislocations and disclinations (or better condensation of 'defects'). Upon increasing the rotational stiffness, disclinations become expensive and at intermediate coupling constants a dislocation-only condensate will emerge. These are the nematic superfluids which are at the focus of this paper.

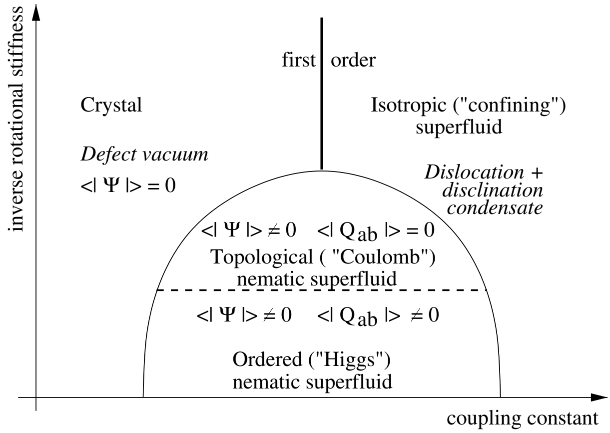

Fig. 6. A refinement of the phase diagram Fig. 5 based on the dual disorder field theory described in this section. The nematic states are associated with Bose condensate of dislocations having a primary order parameter $\langle|\Psi|\rangle$. In addition, the states are characterized by a secondary director order parameter $\langle|Q_{ab}|\rangle$ which is referring to the breaking of space-rotational invariance. In the dislocation condensate the director might (ordered nematic) or might not (topological nematic) acquire an expectation value. In both cases the disclinations are massive, condensing only at the phase boundary with the isotropic superfluid. This offers an explanation of the states first identified by Lammert et al. [3] using abstract gauge theoretical arguments.

nematic- and the isotropic fluids *superfluids*. In the existing literature it is an automatic reflex to associate the superfluid to the presence of interstitials coexisting with the topological condensates (e.g. [17]), and we are considering the situation where interstitials are explicitly forbidden. A main aim of this work will be to prove that the pure dislocation condensate is at the same time a conventional superfluid (Sections 6 and 9). The other novelty is associated with the precise nature of the nematic regime. Again under the condition that interstitials are excluded, we will find two distinct nematic phases (Fig. 6): a phase displaying topological nematic order which does not break rotational invariance, and a phase which does break rotational invariance. The latter is, in a sense which will be explained, a 'quantum smectic.' However, it is distinctly different from the quantum smectic or 'sliding phases' discovered by Kivelson et al. [20,26]. The latter occur under special strong Umklapp conditions and their fluid characteristics reflect the motions of interstitials. Our quantum smectic originates in topology and the special dynamical condition of glide, introduced in the next section.

3. Duality in elasticity: the fundamentals

After these preliminaries, we now arrive at the core of our paper. In this section we will introduce the basic ingredients. To address the quantum dynamics in $2+1$D we need a systematic mathematical formalism and, in this regard, we rely heavily on

Kleinert's treatise [2]. The first, and most crucial step is that we need to dualize the familiar dynamical phonons (Goldstone modes associated with crystalline order) into a gauge-theoretical 'stress-photon' representation. Instead of the familiar phonon fields expressed in infinitesimal displacements one finds instead that the motions are parameterized in terms of non-compact $U(1)$ gauge fields carrying a number of distinct flavors referring to Burgers vector components. We are actually not aware of an explicit treatment of the theory of stress-photons in the context of $2+1$D quantum elasticity and we will dedicate Sections 4 and 5 to a detailed analysis of this counterintuitive affair. The advantage of this unfamiliar representation is that the dislocations act as sources for the stress photons, and the description of interacting dislocation matter turns into a straightforward extension of electromagnetism (Section 3.2). Accordingly, the fluid state realized as the dislocations spontaneously proliferate turns into a Bose-condensate of particles carrying stress charge and this will turn out to be a close analogy of the electromagnetic Meissner–Higgs state (Section 3.3). A novelty is that the dislocation currents and stress gauge fields carry the 'Burgers flavors.' In Section 3.3 we present a first main result: using general symmetry principles we derive the disorder field theory describing the collective behaviors of dislocation matter. Conventional nematic order (breaking of space-rotational symmetry by a director order parameter) follows straightforwardly. However, depending on microscopic conditions, also a nematic state is possible which is characterized by a mere topological order. This is discussed in Section 3.4 where we establish the connection with the Ising gauge-theory of Lammert et al. [3]. Last but not least, in Section 3.5 we introduce the final ingredient which will play a central role in the quantum theory: the glide principle. We will present in this section the proof for the surprising fact that the glide constraint is equivalent to the requirement that translational symmetry breaking leads to the emergence of shear rigidity, leaving compression rigidity unaffected. After the technical Sections 4 and 5, these various ingredients will be brought together in Section 6 where the true nature of the nematic duals will be exposed.

3.1. General considerations

We will demonstrate that under the assumptions discussed in the previous section, the transition between the elastic medium and the nematic superfluids is governed in the scaling limit by a straightforward extension of the well known Abelian–Higgs duality (e.g., see [7] and Appendices A and B). This turns out to be a feat of symmetry. Abelian–Higgs duality is associated with the quantum phase transition associated with a simple phase degree of freedom (global O(2) symmetry), and it demonstrates that the $U(1)$ Meissner phase in $2+1$D is dual to XY order. Among others, it describes the quantum phase transition between the superfluid and the Bose Mott-insulator. The theory of quantum elasticity is, because of its tensorial nature, far more complicated. However, these complications are not essential so long as disclinations are avoided. Dislocations are just about translations and their Abelian nature makes possible that the theory describing the proliferation of dislocations simplifies dramatically, to turn into a Abelian–Higgs duality in disguise.

Goldstone modes are the excitations implied by order, and the topological excitations are associated with the destruction of this order. The Abelian–Higgs duality (see Appendix A) rests on a simple transformation showing that the vortices associated with O(2) (*XY*) long range order in 2 + 1D can as well be viewed as particles carrying electrical charge, interacting with each other via $U(1)$ gauge bosons ('photons') parameterizing the long range interactions between the vortices mediated by the spin waves (Goldstone bosons). Hence, non-compact electromagnetism in 2 + 1D can be seen literally as just a reparametrization of the physics of global O(2) (dis)order. Starting out with a small coupling constant in the global O(2) 'universe,' vortices are massive excitations appearing as vortex–anti-vortex pairs (closed loops in space–time). Upon increasing the coupling constant these loops grow until they become as large as the size of the system. At this point, the vortices become real excitations proliferating in the vacuum. This corresponds with the quantum phase transition to the disordered, symmetrical state of the O(2) system. However, because in the dual 'universe' vortices are just bosonic particles carrying an 'electrical' charge mediated by photons, nothing prohibits the dual particles to Bose condense, and this corresponds with a Meissner state because the dual system is $U(1)$ gauged. The general lesson is that order and disorder are just a matter of viewpoint. An observer having machinery allowing him to measure the *XY* degrees of freedom will insist that his/her spins break symmetry spontaneously at low coupling constant with symmetry restored at large coupling constant. Alternatively, an experimentalist not knowing better that electromagnetism exists, will insist that in his/her universe superconducting order sets in at large coupling constant, getting destroyed upon decreasing the coupling constant. We refer the reader unfamiliar with these notions to Appendices A and B where we present a synopsis of this duality transformation.

Superficially, the theory of (isotropic) quantum elasticity, Eq. (9), is quite similar to the O(2) quantum non-linear sigma model of Appendix A (Eq. A.2), with the displacement fields u^a of the former taking the role of the phase fields ϕ of the latter. A main difference is in the 'upper' labels a, expressing that particles can move both transversally and longitudinally relative to the propagation direction of the phonon, while *XY* spins can only precess. However, considering matters more closely, the theory is much richer. The reason is symmetry. The symmetry principle behind the theory of elasticity is the breaking of the Euclidean group down to a lattice group (in 2 + 1D elasticity: $E(2) \times O(2)$, $E(2)$ is the 2D Euclidean group, O(2) imaginary time dimension). The latter correspond with an infinite group formed from discrete lattice translations- and rotations. In contrast to the *XY* problem, we are dealing here with the complications of non-abelian symmetry.

The disclinations are the topological defects exclusively associated with translations, and translations commute. Disclinations carry the 'magnetic' (dual) charges associated with the rotations and the difficulties coming from the non-abelian nature of the underlying symmetry become central when disclinations start to play a dynamical role. Assuming that the disclination stay massive, the vacuum sector can be described entirely in terms of the dislocations, and since these involve only translations the theory is of an abelian nature and tractable in principle. The theory is still more complicated than phase dynamics because dislocations carry vectorial charges (the

Burgers vectors) but these do not cause essential complications. In essence, under dualization Burgers vectors turn into labels 'flavoring' an abelian gauge theory with a structure similar to the phase dynamics theory. Disclinations correspond with massive particles appearing as excitations relative to the nematic vacua, and their physics is in principle remarkably complex. The richness associated with non-abelian duality structures becomes manifest on this level [34] and we will leave this for future study.

3.2. From phonons to stress photons

Here we will present the first step of the duality: re-parameterizing the phonons in stress-photons and identifying the dislocations as sources. This is discussed at great length in [2], and we just summarize here the main steps. These follow the same pattern as the Abelian–Higgs duality summarized in Appendix A.

Starting with the action Eq. (9), we should take into account the defects that might be present. The presence of defects renders the displacement field configurations to become multivalued. In analogy with vortices, these can be made explicit. To this end we introduce *plastic strain tensors*,

$$w^a_{\mu,P} = \partial_\mu u^a_{MV}, \tag{19}$$

where u^a_{MV} singles out the multivalued (or 'singular') configurations. The elastic energy can only depend on the difference between the elastic strains and the plastic strains. Hence, we should insert for the strains in Eq. (9) the total strain,

$$w^a_{\mu,\mathrm{tot}} = w^a_\mu - w^a_{\mu,P}. \tag{20}$$

We now rewrite Eq. (9) in terms of the asymmetric strains $w^a_{\mu,\mathrm{tot}}$, keeping implicit the condition that only the spatially symmetric strains enter the action. Next, we introduce auxiliary fields σ^a_μ and apply the Hubbard–Stratanovich transformation to Eq. (9),

$$\begin{aligned} Z &= \int Du_a \int D\sigma^a_\mu \exp\left[- \int \tilde{\mathcal{L}}[u_a, \sigma^a_\mu]]\, \mathrm{d}\Omega \right], \\ \tilde{\mathcal{L}} &= -\frac{1}{2}\sigma^a_\mu C^{-1}_{\mu\nu ab}\sigma^b_\nu + \mathrm{i}\sigma^a_\mu\left(w^a_\mu - w^a_{\mu,P}\right). \end{aligned} \tag{21}$$

The elastic strains w^a_μ correspond with the single valued displacement field configurations. Accordingly,

$$\begin{aligned} \mathrm{i}\sigma^a_\mu w^a_\mu &= \mathrm{i}\sigma^a_\mu \partial_\mu u^a, \\ &= -\mathrm{i}u^a \partial_\mu \sigma^a_\mu. \end{aligned} \tag{22}$$

The derivative can be shifted as the fields are integrable. We observe that the smooth fields u^a just enter as a Lagrange multiplier. After an integration, a Bianchi identity for the σ fields follows:

$$\partial_\mu \sigma^a_\mu = 0. \tag{23}$$

Thus these fields are conserved, independently, for every upper label a. The σ fields are none other than the familiar stress fields, and the above operation merely corresponds with the standard stress–strain duality. Eq. (23) encapsulates the conservation of stress. By including the time axis ($\mu = \tau$), Eq. (23) automatically represents the equations of motion.

Alternatively, the stress action can be obtained by varying the action with respect to the strains,

$$\delta S = C_{\mu\nu ab} w^a_\mu \delta w^b_\nu, \tag{24}$$

and the stress fields are, by definition,

$$\sigma^b_\nu = \frac{\delta S}{\delta w^b_\nu} = C_{\mu\nu ab} w^a_\mu. \tag{25}$$

Hence, stress can be directly expressed in terms of strain and this will turn out to be quite useful on several occasions. Specializing to the isotropic 2 + 1-dimensional case ($\hbar = c_{\text{ph}} = 1$),

$$\sigma^a_b = 2\mu w^a_b + (\kappa - \mu)\delta_{ab}\left(w^x_x + w^y_y\right), \qquad \sigma^a_\tau = \mu \partial_\tau u^a. \tag{26}$$

A second constraint follows from the fact that only the strains symmetrical in the space-indices enter the strain action. One has to add this as a constraint in the above duality transformation and it follows immediately that the stress fields have to be symmetric in the space-indices,

$$\sigma^x_y = \sigma^y_x. \tag{27}$$

This condition is known in the elasticity literature as the Ehrenfest theorem [11].

Following Kleinert, let us now depart from the classic treatise of elasticity by realizing that, like in the Abelian–Higgs problem, the conservation law Eq. (23) implies that the dynamics can be parameterized in terms of $U(1)$ gauge fields B^a_μ. As compared to the *XY* case, the difference is that these gauge fields now carry an additional 'Burgers flavor' $a = x, y$,

$$\sigma^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda. \tag{28}$$

We stress that these fields are *not* two-forms, but instead one-forms with additional 'internal' degrees of freedom $\sim a$ which suffice to keep track of the translations. In terms of the 'stress photons' B^a_μ, the theory of elasticity turns into a form of electromagnetism 'flavored' by the Burgers labels,

$$\begin{aligned} Z &= \int \mathcal{D}B^a_\mu \delta\left(\partial_\mu B^a_\mu\right) \exp\left[-\int \mathrm{d}\Omega \mathcal{L}^{\text{dual}}_0\left(B^a_\mu\right)\right], \\ \mathcal{L}^{\text{dual}}_0 &= \frac{1}{2}\epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda C^{-1}_{\mu\mu' ab}\epsilon_{\mu'\nu'\lambda'}\partial_{\nu'}B^b_{\lambda'} \end{aligned} \tag{29}$$

excluding the gauge volume in the measure. The Ehrenfest condition Eq. (27) should also be imposed on the gauge fields,

$$\partial_y B^y_\tau - \partial_\tau B^y_y = -\partial_x B^x_\tau + \partial_\tau B^x_x. \tag{30}$$

Up to this point we have just re-parameterized the overly familiar theory of acoustic phonons in a theory describing force carrying gauge bosons. Although it is still a free Abelian gauge theory, one already anticipates that it looks quite different from the familiar phonon representation. The next two sections are devoted to technicalities needed to uncover the physics in this mathematical construction. The advantage becomes obvious considering the singularities.

We have yet to deal with the plastic strain tensors appearing in Eq. (21). These are associated with multivalued displacement field configurations u^a_P which may not be integrated out,

$$\mathcal{L}^1_{\mathrm{dual}} = \mathrm{i}\sigma^a_\mu w^a_{\mu,P} \equiv \mathrm{i}\sigma^a_\mu \partial_\mu u^a_P = \mathrm{i}\epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda \partial_\mu u^a_P = \mathrm{i}B^a_\mu J^a_\mu. \tag{31}$$

The dislocation currents $J^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda u^a$ (Eq. 16) are recognized as sources for the stress gauge fields. In 'stress electromagnetism', the dislocations have the same role as charge particles have in normal electromagnetism, at least in 2 + 1D. Elasticity, on this level, is sufficiently similar to the *XY* problem in 2 + 1D that the essence of Abelian–Higgs duality is still applicable. The Goldstone modes dualize in gauge fields, while the topological defects acquire the role of sources. Elasticity is richer in the regard that the theory is flavored by the Burgers labels, reflecting the fact that translations are more interesting than an internal O(2) symmetry. However, although these make the problem richer, they do not pose a problem of principle.

Another matter is that the above duality is suffering from the shortcoming that the disclination currents are implicit; they just enter as sources for the dislocation currents, Eq. (17). The disclinations can be made explicit in the formalism by using the double curl gauge fields discovered by Kleinert [2]. Schematically, the stress fields are parameterized by,

$$\sigma^a_\mu = \epsilon_{\mu\nu\lambda}\epsilon_{a\nu'\lambda'}\partial_\nu\partial_{\nu'}h_{\lambda\lambda'}, \tag{32}$$

where $h_{\mu\nu}$ are genuine two-forms. It is readily found that these fields are minimally coupled to a source $\sim \mathrm{i}h_{\mu\nu}\eta_{\mu\nu}$ where $\eta_{\mu\nu}$ corresponds to a two-form called 'defect density.' This can be decomposed as

$$\eta_{\mu\nu} = \eta^\Omega_{\mu\nu} + \frac{1}{2}\left[\epsilon_{\lambda\mu a}\partial_\lambda J^a_\nu + \epsilon_{\lambda\nu a}\partial_\lambda J^a_\mu - \epsilon_{\mu\nu a}\partial_\lambda J^a_\lambda\right], \tag{33}$$

where the J^a_μs are the dislocation currents while $\eta^\Omega_{\mu\nu}$ corresponds with the 'full' disclination currents. In terms of these double curl fields the dynamics of disclinations in the dislocation condensate can be directly addressed—see [41] for the Lorentz-invariant case. One infers that at the moment that disclinations start to play a dynamical role there is a need for a different kind of theory. However, imposing the nematicity condition just means that disclinations can be neglected and under these circumstances dislocations are just like flavored vortices.

3.3. The structure of the dual disorder field theory

In the previous paragraphs we introduced a useful formalism for the description of the physics of isolated ('classical') dislocations. Dislocations are sources of kinetic energy. These will occur in any crystal as virtual excitations in the form of closed dislocation–anti-dislocation loops in space–time. Upon increasing the coupling constant, the characteristic dimension of these loops will grow, until a loop blow out will occur, signaling the transition to the quantum fluid state. On the disordered side of the phase transition, one is dealing with quantum matter composed from strongly interacting defects, exhibiting a dual order. In the case of the Abelian–Higgs problem this is just a Bose condensate of vortices exhibiting a dual Meissner effect because of the coupling to the dual photons. Resting on universality, this state is just described by a dual Ginzburg–Landau–Wilson theory describing the 'disorder parameter' dynamics corresponding with the superconducting order parameter expressing the off-diagonal long range order of the vortices. In this section we will derive the form of the universal disorder field theory describing dislocation matter. To the best of our knowledge our treatment is novel; from this point onward we depart from the established wisdoms.

As compared to the vortices, disclination condensates are more complicated than vortex condensates because both the stress gauge fields and the dislocation currents are flavored by the Burgers labels. However, these do not pose a fundamental problem as we will now show. As a consequence of the abelian nature of the translations, the dislocation currents can be factorized,

$$J^a_\mu(\vec{r}) = \delta^{(2)}_\mu(L,\vec{r})n_a, \tag{34}$$

where $\delta^{(2)}_\mu(L,\vec{r})$ is the line delta function specifying the locus of the dislocation worldline in space–time $\vec{r}$, and n^a is the ath component of the Burgers vector. This factorization makes possible to follow the standard strategy to obtain the effective Ginzburg–Landau disorder field-theory describing the tangle of dislocation worldlines in terms of a complex scalar G–L field $\Psi = |\Psi|e^{i\phi}$, with the interpretation that $|\Psi|^2$ corresponds the density of dislocations in the condensate while the phase field ϕ parameterizes the entanglement (see e.g. [7], Appendices A and B).

The minimal coupling between the dislocation currents and the stress gauge fields can be written as (dΩ space–time volume element),

$$S_{BJ} = \int J^a_\mu B^a_\mu \, d\Omega = \int \delta^{(2)}_\mu(L,\vec{r})n_a B^a_\mu \, d\Omega = \int n_a B^a_\mu \, dr_\mu. \tag{35}$$

The canonical momentum follows immediately,

$$P_\mu = p_\mu + n_a B^a_\mu. \tag{36}$$

We observe that the effective gauge field in the Hamiltonian formulation is just $\vec{n}\cdot\vec{B}_\mu$. This simple result is very important. As we are dealing with a line-entanglement problem, the disorder field theory should have the form of a Ginzburg–Landau–Wilson action. The kinetic part of this action should be $\sim P^2$. Upon the substitution with $p_\mu = i\partial_\mu$ this acts on some disorder field Ψ. In case one is dealing with a vector order parameter theory based on an internal symmetry, the disorder field carries itself the vector labels $\sim \Psi^a$ (e.g., the 'soft spin' parameterization of non-linear sigma models). Here the situation is different. The Burgers vectors themselves refer to space and a priori one cannot exclude that the covariant derivatives would explicitly depend on the Burgers labels as well, such that these would act in a non-trivial way on the order parameter field Ψ^a. Eq. (36), however, demonstrates that the canonical momentum does not carry an explicit dependence on the Burgers vectors because of the product with the stress gauge fields. It just depends on the space–time direction μ.

This will become crucial when we develop further the disorder field theory. We will find that the glide constraint (Section 3.5) has the unusual consequences that the dislocation worldline loops are oriented in planes spend by the Burgers vector and the time direction. One could be tempted to think that this would have the effect that the dual order parameter field could become 1D. However, the above argument shows that this is never the case: the ordering field knows about the full embedding space. In Appendix B these matters are analyzed from the loop gas perspective and the outcome is fully consistent with the Landau style derivation of the previous paragraph.

Let us start out assuming that the field acted on by the covariant derivatives is just a simple complex scalar field Ψ. We find for the piece of the disorder field theory involving the gauge fields,

$$S_{LR} = \int \mathrm{d}\Omega \left[\left| \left(\partial_\mu - \mathrm{i} n_a B^a_\mu \right) \Psi \right|^2 + \frac{1}{2} \sigma^a_\mu c^{-1}_{\mu\nu ab} \sigma^b_\nu \right]. \tag{37}$$

In addition, a mass term $\sim m^2|\Psi|^2$ is present. The mass controls the size of the dislocation loops. These have a finite length for $m^2 > 0$, while the loops 'blow out' when $m^2 < 0$. Finally, a term $\sim w\Psi^4$ has to be added describing the short range interactions [7]. We observe that $|J^a_\mu| \sim |\Psi| n_a$ with the implications that the Burgers vectors enter the disorder field theory like $m^2|\Psi|^2 n^2 \widehat{n_a} \widehat{n_a} + w|\Psi|^4 n^4 (\widehat{n_a})^2 (\widehat{n_b})^2$. In the absence of disclinations one has however to impose the condition that in a finite volume of space no 'ferromagnetic polarization' of Burgers vectors can occur: Burgers vectors have to be locally anti-parallel. The reason for this *local* charge neutrality condition is topological. A finite uniform 'Burgers polarization' corresponds with a disclinations and these we assumed to be massive. The local 'Burgers vector neutrality' implies that the only allowed invariants are scalars and traceless tensors,

$$Q_{ab} = |n|^2 \left(\hat{n}_a \hat{n}_b - \frac{1}{2} \delta_{ab} \right). \tag{38}$$

These describe directors in 2D space, 'O(2) vectors with head and tails identified.' Because Q_{ab} is traceless, the mass term cannot depend on it. It follows that the disorder field theory has the form,

$$S_\Psi = S_{LR} + \int \mathrm{d}\Omega \left[m^2_\Psi |\Psi|^2 + w_\Psi |\Psi|^4 - r|\Psi|^4 Q_{ab} Q_{ba} \right]. \tag{39}$$

The 'director sector' associated with the Q_{ab}s should have its own dynamics. When dislocations with anti-parallel Burgers vectors collide they might annihilate pairwise and re-emerge with a different overall director orientation. The director does not commute with the Hamiltonian and is therefore subjected to fluctuations. These can be incorporated by adding,

$$S_Q = \int \mathrm{d}\Omega \left[\partial_\mu Q_{ab} \partial_\mu Q_{ba} + m^2_Q Q_{ab} Q_{ba} + w_Q Q_{ab} Q_{bc} Q_{cd} Q_{da} \right] \tag{40}$$

corresponding with a non-linear sigma model for the traceless tensors Q_{ab} in a soft spin representation. Since directors can only exist when the dislocations have proliferated, one should impose that $m_Q^2 > 0$ although the total mass $m_Q^2 - r|\Psi|^4$ can become negative ($r > 0$ should be imposed). Notice that for the 'semi-circle directors' of relevance to 2D space cubic invariants are forbidden while they are allowed for the projective plane director order parameters of relevance to 3D space.

Using just the duality notion and symmetry principles, we have derived a description of nematic liquid crystalline order which is radically different from the standard interpretations found in textbooks on liquid crystals. In spirit it is of course quite similar to the KTNHY-theory of melting in two classical dimensions [1]. It can be viewed as a generalization of KTNHY to $2+1$ quantum- or 3 classical dimensions although in some essential regards our formulation is more complete. Most importantly, we arrive at a physical interpretation of the 'topological nematic order' identified by Toner, Lammert, and Rokshar on basis of abstract gauge theoretical arguments. This has been overlooked by KTNHY, and the possible existence of such a state implies that the standard order parameter theory of nematic states is in essential regards incomplete and potentially misleading. Let us explain this in more detail.

3.4. The topological- or 'Coulomb' nematic

The order parameter theory for nematics was established in the early 1970s by de Gennes and Prost [42], and it is of precisely the form Eq. (40). However, the physical interpretation is entirely different. The physical perspective is that of kinetic gas theory. A gas of rod-like molecules is considered and it is assumed that potential energy is gained at collisions when the long axis of the molecules line up. The Q's parameterize this size anisotropy and Eq. (40) follows after averaging. The macroscopic director order parameter reflects, in a direct way, the properties of the microscopic constituents and the director is viewed as an internal symmetry (the orientations of the molecules) detached from the symmetries of space–time. In the language of this paper, it is a theory addressing the degrees of freedom of the interstitials.

Here we take the opposite perspective. The starting point is the state breaking the space symmetries maximally (the crystal, elastic medium) and disordered states are derivatives of this maximally ordered state. These correspond with condensates of the topological excitations associated with the maximal order. Since crystalline order involves the breaking of the Euclidean group down to a lattice group, the charges of the topological excitations (Burgers-, Franck vectors) relate to the symmetries of space itself, and the order parameters characterizing the partial orders refer to the breaking of the space symmetries, and not to some internal symmetry. In this perspective liquid crystalline orders correspond with dual orders in terms of the disorder parameters and these allow for a richer structure. KTNHY [1] follow, in essence, the same path as we do but their attention is limited to the breaking of manifest symmetries (algebraic translational and algebraic rotational order in the 'solid' and 'hexatic' phases, respectively). There is, however, yet another phase which can only be understood in terms of the disorder parameter structure and is best called a state characterized by topological nematic order.

As we already repeatedly emphasized, a state should be called nematic when it is a liquid characterized by massive disclinations carrying quantized, sharply defined, Franck vectors. This topological definition is more general than the usual definition stating that a nematic state is one which is translationally invariant while it breaks spontaneously rotational symmetry. It follows immediately from Eqs. (39) and (40) that a state exists which is *not* breaking rotational invariance while it is a nematic in the topological sense.

Let us consider Eqs. (39) and (40) in more detail. This theory only makes sense in the parameter regime where Ψ is the primary order parameter. The dislocation loops have first to blow out, and only when free dislocations occur at a finite density it is meaningful to consider the order of their Burgers vectors. This implies that $w_\Psi - r|Q|^2 > 0$ to assure stability, while $m_Q^2 > 0$ in order to prevent the Qs to condense before the Ψs have condensed. Under these conditions, the directors will always follow parasitically the Bose-condensation of the dislocations. In this regime, the effect of the mode-coupling will be to renormalize the mass of the director field into $m_{Q,\mathrm{eff}}^2 = m_Q^2 - r|\Psi|^4$. The squared amplitude $|\Psi|^2$ will grow like $[(g - g_c)/g_c]^{2\beta}$ as function of the deviation from the critical coupling constant g_c with β being the order parameter exponent, and it follows that $m_{Q,\mathrm{eff}}^2 = m_Q^2 - r[(g - g_c)/g_c]^{4\beta}$. If $m_Q^2 = 0$, the director order parameter switches on parasitically, directly upon entering the the dislocation Bose condensate. On this level, the transition appears as a 3D *XY* transition from the solid to a conventional nematic. However, when $m_Q^2 > 0$, the dimensionless coupling constant $(g - g_c)/g_c$ has to exceed a critical value before the effective director mass turns negative at a coupling constant g_c'. Hence, in between g_c and g_c', a state exists which does not exhibit director order (rotational symmetry breaking). At the same time, it is not a normal isotropic fluid because disclinations are massive. We conclude that a nematic state exists with an order which can only be measured by non-local means, by the insertion of disclinations, and it therefore corresponds with a truly topological order. One expects such a state to become stable at relatively small rotational stiffness and the topology of the phase diagram should be as indicated in Fig. 6.

Obviously, such a state cannot be imagined starting from a gas of rods. In order to make it work, one needs that the director is a composite of vectors, where the vectors can have a life of their own, while at the same time it is necessary that one can identify disclinations without referral to director fields. This is unimaginable, starting from the gas limit. Nevertheless, this topologically ordered state was identified before, using abstract but elegant arguments based on gauge invariance. Lammert et al. [3] realized that the theory Eq. (40) is characterized by an Ising gauge symmetry: the action is invariant under $\vec{n} \rightarrow -\vec{n}$ because the physical meaningful entity is $Q \sim n^2$. They argued that the theory can be generalized by making this gauge symmetry explicit in terms of the theory of rotors minimally coupled to Z_2 gauge fields. This can be regularized on a lattice as,

$$S = \int \mathrm{d}\Omega \left[-J \sum_{\langle ij \rangle} \sigma_{ij} \vec{n}_i \cdot \vec{n}_j - K \sum_{\mathrm{plaq}} \sigma\sigma\sigma\sigma \right] \tag{41}$$

describing rotors $\vec{n}$ living on the sites of the square lattice coupled through links where Ising valued (± 1) fields σ_{ij} live, determining the sign of the 'exchange' interactions. The second term is the Ising Wilson action corresponding with the product of the Ising gauge variables encircling the plaquettes. This theory is invariant under the gauge transformations corresponding with flipping all σs departing from a given site and simultaneously multiplying the vector on the same site by -1. It is well known that theories like Eq. (41) have three phases: (1) The Higgs phase corresponding with ordered rotors turning into directors under gauge transformations, i.e., just the ordered nematic. (2) The confining phase, where the rotors are disordered, while also the gauge fluxes have proliferated. This is in one to one correspondence with the state where conventional disclinations have condensed, i.e., the isotropic fluid. (3) The Coulomb phase, which is the surprise. The rotors are disordered but the gauge fluxes (or 'visons') are still massive. This state carries the topological order.

Obviously, such a Coulomb phase cannot be imagined starting from the conventional 'gas of rods' perspective. However, in terms of the elastic duality, it acquires a simple physical interpretation. The Ising gauge fluxes are just the π disclinations associated with the solid, and the vectors are in one to one correspondence with the Burgers vectors. The gauge symmetry is associated with the very physical local constraint coming from the Burgers vector neutrality condition. Surely, besides the 'conventional' ordered nematic/Higgs- and isotropic fluid/confining phase there is nothing against a phase where Burgers vectors refuse to order while the 'crystal' disclinations are still massive.

This Coulomb phase will be at center stage in much of the remainder of this paper for technical reasons: it is a simpler state than the ordered nematic, and the properties of the latter are easily deduced from those of the former.

3.5. Dynamics and the glide principle

The next novelty has everything to do with the 'quantum' in the title of this paper. Up to this point it could have been as well a treatise on 3D classical elastic matter, except for some symmetry adjustments. Quantum physics implies that space and time have to be considered simultaneously and starting from non-relativistic matter (see [41] for the relativistic extension) dislocations are subjected to a fundamental, yet independent *dynamical* constraint. In classic elasticity theory [12] this is known as the *glide principle*: dislocations can only move in the direction of their Burgers vector. In the present field-theoretic formulation, this dynamical principle has remarkably far reaching implications.

We already encountered twice the condition that the fields relevant to elasticity are symmetric tensors in the space indices. This is true for the strain- and stress fields, Eqs. (2) and (27), but it should also be imposed on the dislocation currents,

$$J^x_y = J^y_x. \tag{42}$$

This is less obvious than one might think at first. It is an independent constraint and it is readily seen that it cannot be deduced from the symmetry of stresses and strains. In the dynamical (quantum) context, it can be viewed as an additional condition descending from the ultraviolet, needed to keep the theory recognizable as a theory derived from the microscopic breaking of translational invariance. Let us present the proof that the glide condition is needed to ensure that the dislocations are the disorder operators associated with the restoration of translational symmetry.

Eq. (42) has the implication that dislocations move *exclusively* in the direction of their Burgers vector. This can be easily seen by rewriting the dislocation currents in terms of δ functions acting on the time slice, $\vec{n}$ is the Burgers vector, $\vec{\bar{r}} = (x, y)$ the position of the dislocation on the time slice,

$$\begin{aligned} J_b^a &= \epsilon_{b\nu\lambda}\partial_\nu\partial_\lambda u_P^a = \delta_b^{(2)}(L)n_a \equiv n_a \int_L \mathrm{d}\tau' \frac{\mathrm{d}\bar{r}_b}{\mathrm{d}\tau'} \delta^{(3)}(\vec{r} - \vec{\bar{r}}) \\ &= n_a \int_L \mathrm{d}\tau' \dot{\bar{r}}_b(\tau')\delta^{(2)}(\vec{r} - \vec{\bar{r}})\delta^{(1)}(\tau - \tau') = n_a \dot{\bar{x}}_b(\tau)\delta^{(2)}(\vec{r} - \vec{\bar{r}}(\tau)) \\ &\equiv n_a v_b \delta^{(2)}(\vec{x}_{1,2} - \vec{\bar{x}}_{1,2}), \end{aligned} \tag{43}$$

where $\vec{v} = (v_x, v_y)$ is the dislocation velocity and the subscripts in $\vec{x}_{1,2}$ refer to the spatial planar projection of the general space–time coordinate. It immediately follows that

$$J_y^x - J_x^y = (n_x v_y - n_y v_x)\delta^{(2)}(\vec{x}_{1,2} - \vec{\bar{x}}_{1,2}) = 0, \tag{44}$$

implying that $\vec{v} \times \vec{n} = 0$, meaning that the dislocation velocity is finite only in the direction parallel to the Burgers vector $\vec{n}$.

Starting from the crystal lattice it is easily seen why this condition has to be imposed. This is just the usual argument found in the classic elasticity textbooks [12], becoming precise in the zero temperature limit. As discussed, to have a meaningful field theory, we have to insist that the interstitial degrees of freedom are virtual. This implies that the 'elementary' particles constituting the solid cannot be transported. Consider the dislocation motions on a microscopic scale. The dislocation with unit Burgers vector corresponds with the endpoint of a row of 'atoms' (in 2 + 1D) of half-infinite length. The 'glide' motion in the direction of the Burgers vector (Fig. 7) is still possible, even under the condition that the elementary matter cannot flow: cut the row of 'atoms' adjacent to the dislocation into a dislocation–anti-dislocation pair and annihilate the original dislocation with the newly created anti-dislocation. The effect is that the dislocation has moved over one lattice spacing. This requires only microscopic motions of individual particles and is therefore not violating the no-interstitial constraint. The 'climb' motion in the direction perpendicular to the Burgers vector is a different matter. In this case, the length of the half-infinite row of atoms has to increase or decrease and this necessarily involves the presence of delocalized interstitial particles. As these are not present, climb motion is impossible. At finite temperatures, interstitials are always present. The glide condition is thereby no longer rigorous, although it is still invariably true that in real solids the conservative glide motions are much easier than the climb motions, driven by the diffusion of interstitials.

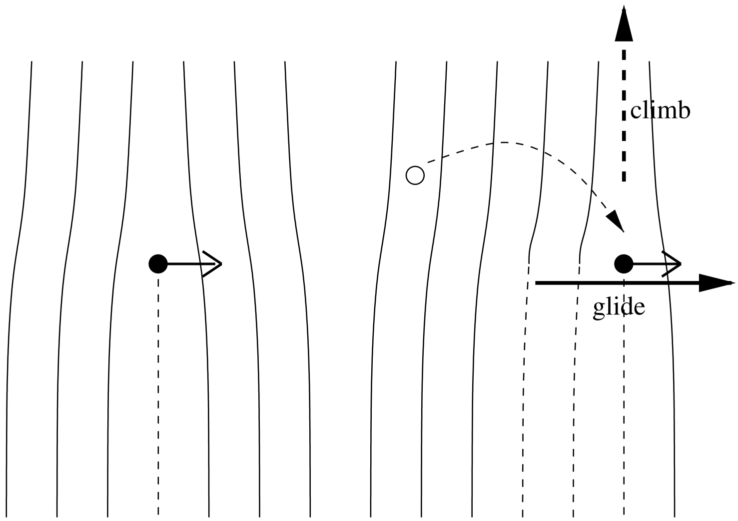

Fig. 7. Even in the complete absence of interstitials (a requirement for the validity of the field theory), dislocations can propagate easily in the direction of their Burgers vector ('glide motion'): cut the neighboring row of atoms, move over the 'tail' and invoking only local atomic motions the dislocation has been transported. In order to move in a direction perpendicular to the Burgers vector, the 'row of atoms' has to become longer and this is only possible in the presence of delocalized interstitials. In the field theory, the glide constraint acquires a central role.

The theory of elasticity describes two distinct rigidities: shear- and compression rigidity. Assuming that the theory describes the long wavelength physics associated with crystalline order, the capacity to carry shear forces is a *consequence* of the breaking of translational invariance. At the same time, compression rigidity is a much more general property which does not require the breaking of translational invariance. Also fluids and gases carry pressure.

Dislocations are the topological excitations which are exclusively associated with the restoration of translational symmetry. Accordingly, dislocations can only interact with the *shear* components of the stresses carried by the medium. By principle, dislocation currents cannot couple to the components of the stress gauge fields B responsible for compressional stresses and the relevant charge in the stress-gauge field formalism should vanish. If this charge would be finite, it would lead to the absurdity that the fluid only carries short range compression forces. It is straightforward to prove that the vanishing of the couplings between dislocations and compressive stresses is identical to the requirement that the space-like dislocation currents are symmetric tensors, Eq. (42).

In full generality, the action of a medium carrying exclusively compression rigidity is,

$$S = \frac{1}{\hbar} \int \mathrm{d}^2 x \mathrm{d}\tau \left[\frac{\kappa}{2} \left(w_x^x + w_y^y \right)^2 + \frac{\rho}{2} \left(\left(w_\tau^x \right)^2 + \left(w_\tau^y \right)^2 \right) \right]. \tag{45}$$

Using Eq.(26), we find for the stress fields

$$\begin{aligned} \sigma^x_x &= \sigma^y_y = \kappa\left(w^x_x + w^y_y\right), \\ \sigma^x_y &= \sigma^y_x = 0, \\ \sigma^a_\tau &= (\rho/2)\partial_\tau u^a. \end{aligned} \tag{46}$$

As compared to the general case, this involves a number of extra constraints. Expressing the stress fields in terms of the stress gauge fields via $\sigma^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda$ the constraints Eq. (46) are uniquely resolved by the 'compression gauge': all Bs are zero except for $B^x_y = -B^y_x = \Phi$, i.e., compression is governed by a single scalar field Φ. The stress fields become $\sigma^x_\tau = \partial_x\Phi, \sigma^y_\tau = \partial_y\Phi, \sigma^x_x = \sigma^y_y = -\partial_\tau\Phi$. Notice that compression involves a single space-like gauge field (Φ) which is time-like (i.e., like the scalar field in electromagnetism).

It follows immediately that the dual action becomes, keeping all units explicit,

$$\begin{aligned} S &= \hbar\int \mathrm{d}\Omega\left[\frac{2}{\kappa}(\partial_\tau\Phi)^2 + \frac{1}{\rho}\left((\partial_x\Phi)^2 + (\partial_y\Phi)^2\right)\right] \\ &= \frac{2\hbar}{\kappa}\int \mathrm{d}^q\,\mathrm{d}\omega\left(\omega^2 + \left(\frac{\kappa}{\rho}\right)q^2\right)|\Phi|^2. \end{aligned} \tag{47}$$

This clearly describes a compression mode. Consider now the coupling to the dislocation currents in the compression gauge,

$$\mathrm{i}J^a_\mu B^a_\mu = \mathrm{i}\left(J^x_y B^x_y + J^y_x B^y_x\right) = \mathrm{i}\left(J^x_y - J^y_x\right)\Phi. \tag{48}$$

This proves that the the dislocation current tensor has to be symmetric in the space indices to satisfy the fundamental requirement that compression does not a carry a gauge coupling to the dislocation currents! Remarkably, in this field theoretic formulation the kinematical glide condition follows from symmetry principle: dislocations should not climb in order to satisfy the condition that translational symmetry breaking is associated exclusively with shear rigidity.

Anticipating that glide will play a central role in the remainder, one can already infer the sense in which the field-theoretic dual can never become literal in condensed matter systems. The culprit is the incapacity of the field theory to keep track of the interstitials. The interstitials interfere in a catastrophic way, because these destroy the glide condition. In real condensed matter systems interstitials cannot be avoided in the nematic quantum fluids. When dislocations are present at a finite density, the probability that dislocations 'collide', i.e., approach each other to microscopic distances, becomes finite. At collisions, the constituent particles can tunnel between dislocations causing the 'rows of atoms' to become shorter and longer (Fig. 8), thereby liberating climb motions. Climb is the same as interstitial transport. Obviously, just releasing climb in the field theory would lead to the absurdity that compression rigidity would vanish, and one has somehow to dress up the theory with separate fields describing the interstitials. At present it is unclear how to construct such a theory.

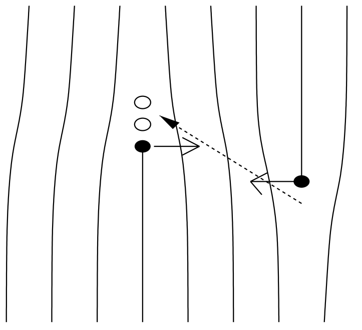

Fig. 8. In condensed matter systems, interstitials cannot be avoided at the moment one enters a fluid state. The reason is that dislocations have a finite probability to collide. At a collision particles can tunnel from one dislocation to the other, thereby liberating climb motions. These processes are of relevance for the long wavelength physics, with the effect that the field-theoretic fluids discussed in this paper cannot be literally realized.

4. The physical fields: helical projection and linear polarization

In the previous section we have collected all necessary ingredients to further develop the theory. However, at first sight it appears as a quite complicated affair. In total we are dealing with six stress gauge fields B^a_μ, while gauge invariance implies six dislocation currents J^a_μ as well. In terms of these gauge fields the dual action acquires a quite complicated form, further complicated by the various constraints. The gauge transversality condition implies that two out of the six fields are unphysical anyhow. In addition, the Ehrenfest symmetry constraint has to be satisfied by the Bs, while the glide condition acts as an extra constraint on the Js. In total, the theory is about three *physical* stress gauge fields. Since the dislocation currents describe the singularities in the physical fields, there are three dislocation currents associated with the physical fields, while one of these three dislocation currents is unphysical because of the glide constraint.

In order to isolate the physical content of the theory it is convenient to employ helical projections. This is discussed at length by Kleinert for the 3D isotropic theory [2]. Helical projections are well known from, e.g., quantum electrodynamics. Transform the theory to momentum space and consider Dreibeins (in 3D) $e^{(\alpha)}$, $\alpha = 0, 1, -1$, with their (0) component parallel to the propagation direction, while the (± 1) components represent left- and right circularly polarized transversal 'photons.' By neglecting the (0) components of the projected gauge fields transversality is imposed ('transversal gauge'). As discussed by Kleinert, the tensorial stress gauge fields in the 3D isotropic theory can be decomposed into $S = 0$ (compression) and $S = 2$ (shear) helical components. In the present $2+1$D case, Lorentz invariance is badly broken due to the 'anomalous' time axis and the helical projections have to be modified accordingly. This space–time anisotropy is reflected in the 'upper' Burgers indices a of the stress-gauge field tensors B^a_μ referring exclusively to spatial directions but also in the 'lower' space–time indices μ. As we will now demonstrate, there appears to be a unique projection satisfying all constraints, putting the stress photons on 'normal coordinates.'

To put matters in perspective, let us first shortly review the standard helical projection. Consider a $U(1)$ gauge field A_μ in 2 + 1D with an Euclidean Maxwell action $S \sim F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the gauge constraint $\partial_\mu A_\mu = 0$. Fourier transform to (Matsubara) frequency–momentum space $\mathbf{p} = (q_x, q_y, \omega)$ and introduce the unit vector $\hat{p} = \mathbf{p}/|p|$. The action becomes $S \sim |p|^2(\hat{p}_\mu A_\nu - \hat{p}_\nu A_\mu)(\hat{p}_\mu A^*_\nu - \hat{p}_\nu A^*_\mu)$ with gauge constraint $\hat{p}_\mu A_\mu = 0$. The helicity basis is constructed as follows: choose an orthogonal set of three basis vectors (Dreibein) $\mathbf{e}^1, \mathbf{e}^{-1}, \mathbf{e}^0$ in the subspace of a fixed momentum $\hat{\mathbf{p}}$ such that $\mathbf{e}^0 = \hat{\mathbf{p}}$, coinciding with the propagation direction of the photon. Lorentz invariance turns into the Euclidean group $E(3)$ after analytic continuation and this is wired in by constructing a spin-one representation,

$$\begin{aligned} \mathbf{e}^{(+1)}(\hat{p}) &= \frac{1}{\sqrt{2}}(\mathbf{e}^1 + \mathrm{i}\mathbf{e}^{-1}), \\ \mathbf{e}^{(-1)}(\hat{p}) &= -\frac{1}{\sqrt{2}}(\mathbf{e}^1 - \mathrm{i}\mathbf{e}^{-1}), \\ \mathbf{e}^{(0)}(\hat{p}) &= \mathbf{e}^0 = \hat{\mathbf{p}}. \end{aligned} \tag{49}$$

Define the projection matrices $P^{(h)}$

$$\begin{aligned} &P^{(h)}_{\mu\nu}(\hat{\mathbf{p}}) = e^{(h)}_\mu(\hat{\mathbf{p}})e^{(h)}_\nu(\hat{\mathbf{p}}), \\ &\sum_h P^{(h)}_{\mu\nu}(\hat{\mathbf{p}}) = \delta_{\mu\nu}, \\ &P^{(h)}_{\mu\lambda}(\hat{\mathbf{p}})P^{(h)}_{\lambda\kappa}(\hat{\mathbf{p}}) = P^{(h)}_{\mu\kappa}(\hat{\mathbf{p}})\delta_{\mu\kappa}(\hat{\mathbf{p}}), \end{aligned} \tag{50}$$

showing that a vector function like the gauge field $\mathbf{A}$ can be expanded in the helical basis,

$$\begin{aligned} A_\mu(\mathbf{p}) &= \sum_h P^{(h)}_{\mu\nu}(\hat{\mathbf{p}})A_\nu(\mathbf{p}) = \sum_h e^{(h)}_\mu(\hat{\mathbf{p}})A^{(h)}(\mathbf{p}), \\ A^{(h)} &= \sum_\mu e^{(h)}_\mu(\hat{\mathbf{p}})A_\mu(\mathbf{p}) \end{aligned} \tag{51}$$

and the $A^{(h)}(\mathbf{p})$ are the helicity components of A_μ. One infers immediately that the gauge constraint,

$$\sum_\mu p_\mu A_\mu = |p| \sum_\mu e^{(0)}_\mu \sum_h e^{(h)}_\mu A^{(h)} = |p|A^{(0)} = 0. \tag{52}$$

It follows that the longitudinal helical component $A^{(0)}$ corresponds with the unphysical content of the gauge field, while the $A^{(\pm 1)}$ fields are the physical components satisfying the gauge-transversality requirement. Using Eqs. (51) and (52) and the orthonormality of the Dreibeins it is easily shown that the Maxwell action becomes

$$\mathcal{L} \sim |p|^2\left(|A^{(+1)}|^2 + |A^{(-1)}|^2\right), \tag{53}$$

expressing the fact that the physical excitations of the electromagnetic field correspond with left- and right circularly polarized photons.

The above has to be modified in the quantum-elasticity context as the Euclidean invariance $E(3)$ is broken in 2 + 1D space–time to $E(2) \times \mathrm{O}(2)$ due to the special role of the imaginary time axis. However, rotational symmetry is maintained on the time slice, suggesting that a decomposition in purely spatial longitudinal- and transversal components should be useful. This is surely the case for the 'upper' Burgers indices a of the stress gauge fields B^a_μ. Parameterize the spatial momentum vector in terms of a phase $\phi_{\mathbf{q}}$ as $\vec{q} = |q|(\hat{q}_x + \hat{q}_y) = |q|\big(\hat{e}_x \cos(\phi_{\mathbf{q}}) + \hat{e}_y \sin(\phi_{\mathbf{q}})\big)$ and decompose the stress gauge fields in longitudinal- and transversal spatial components of their Burgers flavors (L, T, respectively),

$$B^L_\mu(\mathbf{p}) = \cos\phi_{\mathbf{q}} B^x_\mu + \sin\phi_{\mathbf{q}} B^y_\mu, \qquad B^T_\mu(\mathbf{p}) = \cos\phi_{\mathbf{q}} B^y_\mu - \sin\phi_{\mathbf{q}} B^x_\mu. \tag{54}$$

This requirement imposes a preferred reference frame on the Dreibeins living in space–time, see Fig. (9). We keep the velocity convention $c^2_{\mathrm{ph}} = 2\mu/\rho = 1$ as introduced in Section 2.1, to define space–time 'momentum' as $\mathbf{p} = (\mathbf{q}, \omega) = (q_x, q_y, \omega) = |p|(\hat{q}_x, \hat{q}_y, \hat{\omega}) = |p|e^0(\mathbf{p})$. The space–time longitudinal unit vector $e^0(\mathbf{p})$ can be written in polar coordinates as,

$$e^0(\mathbf{p}) = \begin{pmatrix} 1 \sin\theta_{\mathbf{p}} \cos\phi_{\mathbf{p}} \\ \sin\theta_{\mathbf{p}} \sin\phi_{\mathbf{p}} \\ \cos\theta_{\mathbf{p}} \end{pmatrix}. \tag{55}$$

In non-relativistic electromagnetism problems it is natural to single out the spatial components of the vector potentials $A_{x,y}$ as responsible for magnetic phenomena. Since Lorentz invariance is badly broken in the present case, one should look for some superposition of the $B^a_{x,y}$ fields representing the 'magnetic' sector of the stress gauge fields. These live on the time-slice as the physical fields have to be orthogonal

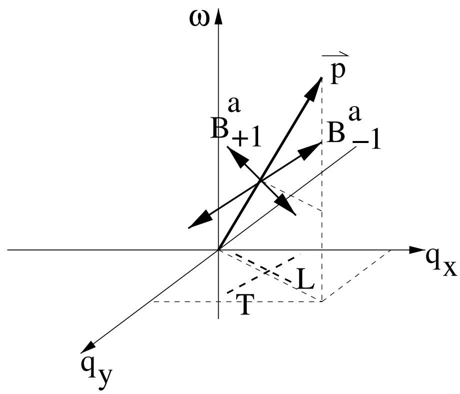

Fig. 9. Explanation of the helical projection used to evaluate the stress gauge fields in terms of their physical content ('stress photons'). The propagation direction of the photon in frequency–momentum space is $\vec{p}$. The Burgers components of the stress-gauge field are decomposed in longitudinal (L) and transversal (T) spatial momentum components, where L is parallel to $\vec{p}$. In order to satisfy the gauge transversality condition, the gauge fields are decomposed in a magnetic component −1 which is parallel to the T spatial component and an electrical component which is orthogonal to both $\vec{p}$ and the magnetic component.

to Eq. (55) the direction of this transverse component has to coincide with the transverse spatial direction which we call '−1' or 'magnetically' polarized,

$$e^{-1}(\mathbf{p}) = \begin{pmatrix} \sin\phi_{\mathbf{p}} \\ -\cos\phi_{\mathbf{p}} \\ 0 \end{pmatrix}. \tag{56}$$

The other transverse component has to be orthogonal to both Eqs. (55) and (56) and this fixes the direction of the remaining +1 or 'electrical' component,

$$e^{1}(\mathbf{p}) = \begin{pmatrix} -\cos\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}} \\ -\cos\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}} \\ \sin\theta_{\mathbf{p}} \end{pmatrix}. \tag{57}$$

Introducing separate 'zweibeins' for the space-like transverse and longitudinal projections,

$$\tilde{e}^{L}(\mathbf{p}) = \begin{pmatrix} \cos\phi_{\mathbf{p}} \\ \sin\phi_{\mathbf{p}} \end{pmatrix}, \qquad \tilde{e}^{T}(\mathbf{p}) = \begin{pmatrix} -\sin\phi_{\mathbf{p}} \\ \cos\phi_{\mathbf{p}} \end{pmatrix}. \tag{58}$$

Now, we have to define rules of change of the basis vectors introduced above in the case of the sign change of the momentum $\vec{p} \to -\vec{p}$ in the frequency–momentum space. Namely, we associate the inversion of momentum $\vec{p} \to -\vec{p}$ with the following change of the angles:

$$\theta_{\mathbf{p}} \to \pi - \theta_{\mathbf{p}}, \qquad \phi_{\mathbf{p}} \to \phi_{\mathbf{p}} + \pi. \tag{59}$$

Thus, we arrive at the following relations between basis vectors sets corresponding to the momenta $\vec{p}$, $-\vec{p}$:

$$\begin{aligned} e^{0}(-\mathbf{p}) &= -e^{0}(\mathbf{p}), \\ e^{-1}(-\mathbf{p}) &= -e^{-1}(\mathbf{p}), \\ e^{1}(-\mathbf{p}) &= e^{1}(\mathbf{p}), \\ \tilde{e}^{L}(-\mathbf{p}) &= -\tilde{e}^{L}(\mathbf{p}), \\ \tilde{e}^{T}(-\mathbf{p}) &= -\tilde{e}^{T}(\mathbf{p}). \end{aligned} \tag{60}$$

These relations have a straightforward implication for our definition of the projected fields in analogy with Eq. (51). Namely, we want to preserve the usual relation $B^{a}_{\mu}(-\vec{p}) = B^{a}_{\mu}(\vec{p})^{*}$ between the Fourier components of the real stress gauge fields also in the $\vec{p}$-dependent basis vectors set defined above in Eqs. (55)–(58). For this purpose, we define projected stress gauge fields $B^{L,T}_{h}$ in the following way:

$$B^{a}_{\mu} = \sum_{E=L,T} \mathrm{i}\tilde{e}^{E}_{a}\left(e^{1}_{\mu}B^{E}_{1} + \mathrm{i}e^{-1}_{\mu}B^{E}_{-1}\right) \equiv \sum_{E=L,T} \tilde{e}^{E}_{a}\left(\mathrm{i}e^{1}_{\mu}B^{E}_{1} - e^{-1}_{\mu}B^{E}_{-1}\right). \tag{61}$$

The multiplication by the imaginary i in Eq. (61) for the basis vectors leads to a change in sign when the momentum $\vec{p}$ is reversed, precisely serving the purpose explained above.

To guide the intuition it is useful to consider what the projections Eqs. (55)–(57) mean in the familiar electromagnetism context. Consider the magnetic- (H^2) and electrical ($E^2 = E_x^2 + E_y^2$) energy densities in energy–momentum space,

$$
\begin{aligned}
H^2 &= |q_x A_y - q_y A_x|^2 = q^2 |A_{-1}|^2 \\
E^2 &= |q_y A_\tau - (\omega/c) A_y|^2 + |-q_x A_\tau + (\omega/c) A_y|^2 \\
&= q^2 |A_{+1}|^2 + (\omega/c)^2 (|A_{+1}|^2 + |A_{-1}|^2).
\end{aligned}
\tag{62}
$$

This demonstrates that this has to do with a linear polarization, such that in the non-relativistic regime $q \gg \omega/c$, the $h = -1$ components are responsible for magnetic phenomena while the $h = +1$ components are in charge of the electrical fields. Although not to be taken literal, we will in the remainder repeatedly refer to 'magnetic-like' and 'electrical-like' stress fields.

The projected fields in Eq. (61) are very convenient. First, by neglecting the '0' components of the B fields, the gauge transversality is imposed. Secondly, the space-like transversal–longitudinal projection has the effect that the Ehrenfest symmetry condition $\sigma^x_y = \sigma^y_x$ (Eq. (27)) acquires a simple form,

$$
\mathrm{i} B^L_{+1} = -\cos(\theta) B^T_{-1} = -\hat{\omega} B^T_{-1}. \tag{63}
$$

This demonstrates that the Ehrenfest condition renders the B^L_{+1} field redundant. Together with the gauge transversality this implies that three physical 'stress photons' exist: B^T_{-1}, B^L_{-1}, and B^T_{+1}. As we will see in the next section, in terms of the projected fields Eq. (61) the stress-gauge field action for the isotropic medium Eq. (29) acquires a quite simple form which is easily diagonalized.

5. 2 + 1D quantum elasticity in gauge fields

Before we turn to the more interesting physics associated with the dislocations, let us first consider how the theory looks like in the absence of these defects. This is just the simple phonon problem discussed in Section 2, now rewritten in terms of the dual degrees of freedom: the stress gauge fields. Although the technology introduced in the previous section simplifies matters to a great extent, the stress-gauge field formulation remains remarkably complicated, at least as compared to the couple of lines of simple algebra needed to identify the phonons in the strain field formalism. We will first derive the action of isotropic elasticity in terms of the stress-photons living on normal coordinates. Although the result has a simple form, it poses a considerable physical interpretation problem. Among others, it carries information in the $\omega \to 0$ limit which seems absent in the strain formulation. In addition, we find stress-photons resembling longitudinal- and transversal phonons, but also a third photon is present which has no counterpart in the strain formalism. At first sight this is quite confusing, but on closer inspection it gives away some interesting insights. Phonons are living in the 'universe' derived from order, while the stress photons belong to the dual 'universe' governed by the disorder fields, the matter associated with the dislocations. The richer, more complex structure associated with the stress photons is needed to understand the physics of the disordered state, as we will see in the next sections. In the ordered state, the stress photons are rather clumsy entities. This will be illustrated in Section 5.2 where we will calculate the elastic propagators starting with the stress photons. This will turn out to be a remarkably long and tedious detour, but this technology will become most useful when dealing with the disordered states.

5.1. Derivation of the stress photon action

In terms of the stress fields σ^a_μ, the dual 'stress-Maxwell' action of isotropic quantum elasticity becomes,

$$S(\nu) = \frac{1}{4\mu}\int \mathrm{d}\Omega\left[\sigma^{x^2}_x + \sigma^{y^2}_x + \sigma^{x^2}_y + \sigma^{y^2}_y - \frac{\nu}{1+\nu}(\sigma^x_x + \sigma^y_y)^2 + \sigma^{x^2}_\tau + \sigma^{y^2}_\tau\right] \tag{64}$$

with the constraint $\sigma^y_x = \sigma^x_y$ to be imposed.

Let us evaluate this action using the projections introduced in the previous section. Substitute $\sigma^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda$, transform to momentum–frequency space, and insert the inverse of Eq. (61). A major simplification occurs in the case $\nu = 0$, such that $\kappa = \mu$. The action simplifies to a Maxwell action, separately for the x and y Burgers flavors,

$$S(\nu = 0) = \frac{1}{4\mu}\sum_{a=x,y}\int \mathrm{d}\Omega\left[\left(\sigma^a_x\right)^2 + \left(\sigma^a_y\right)^2 + \left(\sigma^a_\tau\right)^2\right] = \frac{1}{4\mu}\sum_{a=x,y}\int \mathrm{d}\Omega F^a_{\mu\nu}F^a_{\mu\nu} \tag{65}$$

with $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu$. In terms of the projected fields in frequency–momentum space this simply becomes (compare with Eq. (53)),

$$S(\nu = 0) = \frac{1}{4\mu}\int \mathrm{d}^2q\,\mathrm{d}\omega|p|^2\left(|B^T_1|^2 + |B^T_{-1}|^2 + |B^L_1|^2 + |B^L_{-1}|^2\right). \tag{66}$$

For $\nu \neq 0$ we have to evaluate the compression-only part $\sim (\sigma^x_x + \sigma^y_y)^2$. A straightforward calculation yields

$$\sigma^x_x + \sigma^y_y = \mathrm{i}|p|\left(\mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}\right). \tag{67}$$

Finally, we have to impose the symmetry condition $\sigma^y_x = \sigma^x_y$, and as we already discussed this translates into the condition $\mathrm{i}B^L_1 = -\hat{\omega}B^T_{-1}$ (Eq. 63), so that the B^L_1 field can be eliminated. Collecting, we find for the full action,

$$S = \frac{1}{4\mu}\int \mathrm{d}^2q\,\mathrm{d}\omega|p|^2\left[|B^T_1|^2 + (1+\hat{\omega}^2)|B^T_{-1}|^2 + |B^L_{-1}|^2 - \frac{\nu}{1+\nu}|\mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}|^2\right]. \tag{68}$$

The compression-only term $\sim \nu$ causes a linear coupling of the B^T_1 and B^L_{-1} modes which is proportional to frequency, while it vanishes in the static limit. This coupling is diagonalized by

$$\begin{aligned}\mathcal{B}^T_1 &= \frac{1}{\sqrt{1+\hat{\omega}^2}}\left(\mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}\right),\\ \mathcal{B}^L_{-1} &= \frac{1}{\sqrt{1+\hat{\omega}^2}}\left(-\mathrm{i}\hat{\omega}B^T_1 - B^L_{-1}\right).\end{aligned} \tag{69}$$

The labels of the $\mathcal{B}$ fields are chosen to reflect the $\omega \to 0$ origin of these fields. At finite frequencies these are of course linear superpositions of the 'bare' B fields. Using $|p|^2 = \omega^2 + q^2$ and $\hat{\omega} = \omega/|p|$ we find the diagonal action describing the stress photons B^T_{-1}, $\mathcal{B}^T_1$ and $\mathcal{B}^L_{-1}$,

$$S = \frac{1}{4\mu}\int d^2q d\omega \left[(2\omega^2 + q^2)|B^T_{-1}|^2 + \frac{1}{1+\nu}((1-\nu)\omega^2 + q^2)|\mathcal{B}^T_1|^2 + (\omega^2 + q^2)|\mathcal{B}^L_{-1}|^2 \right]. \tag{70}$$

This is an important result: it is the theory of quantum elasticity in the dual, gauge field, representation. In a magical way, it seems, familiar information has resurfaced. In the terms multiplying the B^T_{-1} and $\mathcal{B}^T_1$ stress photons the dispersion relations of the transverse and longitudinal phonons are recognized. Halfway the calculation this information was completely scrambled. Consider, for instance, the B^T_{-1} stress photon. This mode appears as an eigenmode already in step Eq. (66). However, the unit of velocity is $c^2_{\text{ph}} = 2\mu/\rho$ while the transverse phonon velocity is $c^2_T = \mu/\rho$ (compare Eq. (12)), a factor of two mismatch. In the final result, the correct velocity has re-emerged thanks to the Ehrenfest constraint. By eliminating the B^L_1 field, the propagator of the B^T_{-1} field picks up an additional $\hat{\omega}^2$ factor and this factor takes care of halving the 'natural' velocity c^2_{ph},

$$p^2(1 + \hat{\omega}^2) = \left(\omega^2 + c^2_{\text{ph}}q^2\right)\left(1 + \frac{\omega^2}{\omega^2 + c^2_{\text{ph}}q^2}\right) = \frac{1}{2}\left(\omega^2 + \frac{c^2_{\text{ph}}q^2}{2}\right). \tag{71}$$

It is easy to verify that also in the strain formalism the halving of the transversal velocity finds its origin in the symmetry of the strain tensor.

The $\mathcal{B}^T_1$ clearly propagates like the longitudinal phonon. However, at the same time it is solely responsible for the propagation of stresses through the medium in the high temperature, classical 2D limit. This is easily seen: the -1 fields describe time-like phenomena and disappear with the time axis. The action reduces to

$$S = \int d^2q \frac{q^2}{4\mu(1+\nu)}|B^T_{+1}|^2, \tag{72}$$

which indeed corresponds with the action in the 2D limit. The reader is referred to Appendix C for a derivation of the static 2D action.

The real surprise is the presence of a second time-like field, $\mathcal{B}^L_{-1}$, propagating with the natural velocity c_{ph}. This mode has no counterpart in the strain formulation! A priori there is no reason that the mode content in the stress-gauge field formulation should be the same as in the strain formulation. In the latter one counts the Goldstone mode content, while the meaning of the gauge fields is fundamentally different, since they parameterize the capacity of the medium to mediate forces. For instance, in the Abelian–Higgs duality a single spin wave translates into two physical photons in the dual representation. The specialty in the present context is that the theory of elasticity involves different velocities, and it is just curious that stress photons exist propagating differently from any phonon. The resolution of this puzzle is that these photons do exist in their dual world, but they cannot be made visible using order (Goldstone mode) means. At the same time, when disorder gets in charge this extra photon acquires a meaning on the disorderly side of the duality transformation: as we will see the third photon has everything to do with the fact that the superfluid carries *three* modes. To shed further light on these matters, let us consider the elastic propagators.

5.2. Gauge fields and elastic propagators

In order to appreciate the physical meaning of the stress photons, it is quite helpful to interrogate them with physical means: the elastic propagators. These were already introduced in Section 2 (Eqs. (13) and (14)) and their spectral functions correspond with the dynamical form factors as measured by neutron- and X-ray scattering. From the strain formalism of Section 2, it is obvious that the poles of the spectral functions correspond with the Goldstone modes, the phonons.

How to compute these propagators in a stress gauge fields language? As we discuss in Appendix C, the relations between the strain propagators (S_0 is the strain action),

$$\langle\langle \partial_\mu u^a(\vec{r}_1)|\partial_\nu u^b(\vec{r}_1)\rangle\rangle \equiv \frac{1}{Z}\int \mathcal{D}u^a \left(\partial_\mu u^a(\vec{r}_1)\right)\left(\partial_\nu u^b(\vec{r}_2)\right)\exp(-S_0) \tag{73}$$

and the stress propagators (S_{dual} is the stress action)

$$\langle\langle \sigma^c_\lambda(\vec{r}_1)|\sigma^d_\kappa(\vec{r}_2)\rangle\rangle \equiv \frac{1}{Z}\int \mathcal{D}\sigma^a_\mu \delta\left(\partial_\mu \sigma^a_\mu\right)\sigma^c_\lambda(\vec{r}_1)\sigma^d_\kappa(\vec{r}_2)\exp\left(-S_{\mathrm{dual}}\left(\sigma^a_\mu\right)\right) \tag{74}$$

are simple, but non-trivial,

$$\langle\langle \partial_\mu u^a(\vec{r}_2)|\partial_\nu u^b(\vec{r}_1)\rangle\rangle = \delta(\vec{r}_1-\vec{r}_2)\delta_{\mu,\nu}\delta_{a,b}C^{-1}_{\mu\mu aa} - C^{-1}_{\mu\lambda ac}C^{-1}_{\nu\kappa bd}\langle\langle \sigma^c_\lambda(\vec{r}_1)|\sigma^d_\kappa(\vec{r}_2)\rangle\rangle. \tag{75}$$

Hence, in momentum space these have the form $\langle\langle p_\mu u^a|p_\nu u^b\rangle\rangle \sim C^{-1}_{\mu\mu aa} - C^{-1}_{\mu\lambda ac}C^{-1}_{\nu\kappa bd}\langle\langle \sigma^c_\lambda|\sigma^d_\kappa\rangle\rangle$. The algorithm is therefore as follows: (a) use the stress–strain relations to determine the correspondence $p_\mu u^a = C^{-1}_{\mu\lambda ac}\sigma^c_\lambda$, (b) calculate the stress propagators $\langle\langle \sigma^c_\lambda|\sigma^d_\kappa\rangle\rangle$ exploiting the stress-gauge fields; these become proportional to the transversal stress-photon propagators, (c) subtract the stress propagator from the constant $\sim C^{-1}_{\mu\mu aa}$ and the elastic propagator follows.

We are particularly interested in the elastic (or phonon) propagators Eqs. (13) and (14) referring only to spatial momenta, corresponding with the observables in condensed matter experimentation. The relevant stress–strain relations are in isotropic elasticity,

$$\begin{aligned}
\mathrm{i}q_x u^x &= \frac{1}{2\mu(1+\nu)}\left[\sigma^x_x - \nu\sigma^y_y\right],\\
\mathrm{i}q_y u^y &= \frac{1}{2\mu(1+\nu)}\left[\sigma^x_x - \nu\sigma^y_y\right],\\
\mathrm{i}q_x u^y &= \frac{1}{2\mu}\sigma^y_x,\\
\mathrm{i}q_y u^x &= \frac{1}{2\mu}\sigma^x_y,
\end{aligned} \tag{76}$$

where the two last relations are true under the constraint that $\sigma^y_x = \sigma^x_y$. Using these relations, the calculation of the elastic propagators becomes a remarkably lengthy but straightforward algebraic exercise.

Let us first consider a simple example to illustrate matters: the compression-only case (Eqs. (45) and (47) in Section 3). Using the strain fields, the propagator follows directly from Eq. (45),

$$G = G_L = \frac{2}{\kappa}\frac{q^2}{q^2 + \frac{\rho}{\kappa}\omega^2} \tag{77}$$

keeping the velocity explicit for clarity.

Since $\partial_x u^x + \partial_y u^y = -(2/\kappa)\partial_\tau \phi$ in compression gauge, the stress–strain relation becomes $q_x u_x + q_y u_y = -(2/\kappa)\omega\phi$. The propagator of the 'compression photon' ϕ follows immediately from the action Eq. (47),

$$\langle\langle \phi|\phi\rangle\rangle = \frac{\kappa}{2}\frac{1}{\omega^2 + (\kappa/\rho)q^2} \tag{78}$$

The elastic propagator is given in terms of the compression photon propagator as,

$$G = \frac{2}{\kappa} - \frac{4\omega^2}{\kappa^2}\langle\langle \phi|\phi\rangle\rangle = \frac{2}{\kappa} - \frac{4\omega^2}{\kappa^2}\frac{\kappa}{2}\frac{1}{\omega^2 + (\kappa/\rho)q^2} = \frac{2}{\kappa}\frac{q^2}{q^2 + (\rho/\kappa)\omega^2} \tag{79}$$

and this is the desired result.

Let us now consider the computation of the elastic propagators of the full problem. Consider first the longitudinal propagator. A useful relation is,

$$q_x u^x + q_y u^y = \frac{1-\nu}{2\mu(1+\nu)} p\left(\mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}\right) \tag{80}$$

and it follows immediately that

$$G_L = \text{Const.} - \left(\frac{1-\nu}{2\mu(1+\nu)}\right)^2 p^2 \langle\langle \mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}|\mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}\rangle\rangle. \tag{81}$$

The transverse propagator is more cumbersome and after some lengthy but straightforward algebra we find

$$\begin{aligned} G_T = \text{Const.} - \frac{1}{4\mu^2}p^2\Big(& 2\hat{\omega}^2\langle\langle B^T_{-1}|B^T_{-1}\rangle\rangle - \mathrm{i}\hat{\omega}[\langle\langle B^L_{-1}|B^T_{+1}\rangle\rangle - \langle\langle B^T_{+1}|B^L_{-1}\rangle\rangle] \\ & + \frac{2\nu}{(1+\nu)^2}\langle\langle \mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}|\mathrm{i}B^T_1 - \hat{\omega}B^L_{-1}\rangle\rangle\Big). \end{aligned} \tag{82}$$

Adding these up, we find for the total propagator

$$\begin{aligned} G = \frac{2(\nu+2)}{\mu(\nu+1)} - \frac{1}{4\mu^2}p^2\Big(& 2\hat{\omega}^2\langle B^T_{-1}|B^T_{-1}\rangle - \mathrm{i}\hat{\omega}[\langle B^L_{-1}|B^T_{+1}\rangle - \langle B^T_{+1}|B^L_{-1}\rangle] \\ & + \frac{1+\nu^2}{(1+\nu)^2}\langle B^T_1 + \hat{\omega}B^L_{-1}|B^T_1 + \hat{\omega}B^L_{-1}\rangle\Big). \end{aligned} \tag{83}$$

The general results Eqs. (81)–(83) are an expression of the remarkably convoluted way in which the stress photons relate to observables associated with the orderly side of the duality. Considering the intact elastic medium, these expressions should eventually simplify to the simple phonon propagators Eq. (13), and this is indeed what happens!

Transforming to the diagonal photon representation using Eq. (69), and using the photon propagators following from (70), the propagator Eq. (83) can be computed. Remarkable cancelations occur, and eventually

$$\begin{aligned} G &= \frac{1}{\mu}\left(\frac{2(v+2)}{v+1} - \frac{\omega^2}{\omega^2+q^2/2} - \frac{2\omega^2(1-v)+q^2(1+v^2)}{(1+v)(\omega^2(1-v)+q^2)}\right) \\ &= \frac{1}{\mu}\left[\frac{(q^2/2)}{\omega^2+(q^2/2)} + \frac{q^2}{(q^2/(1-v))+\omega^2}\right]. \end{aligned} \tag{84}$$

We indeed recover the simple phonon propagator Eq. (14).

All together, this is a remarkably tedious calculation, especially as compared to the elementary derivation of the propagators using the strain fields. Is this just an un-necessary detour, or is there a deeper meaning to it? It is actually so that the stress photons parameterize the physics of the medium in a more complete way than the strain phonons do. We already pointed at the $\mathcal{B}_L^{-1}$ stress photon which has no image in the phonon sector. Nevertheless, this photon is physical. In order to parameterize all the forces which can be transmitted in the elastic medium one needs a physical photon for every dimension, meaning that three stress-photons have to exist given that space–time is 2 + 1D. At the same time, in order to explain any outcome of an experiment on the medium using machines constructed from orderly matter (like neutron sources) it is clear that one can get away just knowing about the Goldstone bosons which are counted according to the number of space dimensions (the two phonons). How can this be? The above calculation gives the answer. A first major complication occurs due to the transformation to the $\mathcal{B}$ photons: the $\mathcal{B}_L^{-1}$ propagators acquire zero weight and the neutrons, etcetera, only couple to the longitudinal phonon-like $\mathcal{B}_T^1$ stress photons. In other words, the $\mathcal{B}_L^{-1}$ are genuine modes but experimental physics does not posses the machines to measure them!

This becomes more manifest when dislocations come into play. When these proliferate, the disorderly side of the duality gets in charge, imposing its physics also on the orderly side of the universe where neutron sources are found. As we will see in the next sections, this has the consequence that the third stress-photon will become visible: the order derived superfluid is characterized by three collective modes.

6. Completing the duality: Coulomb nematics as superfluids

In the last two sections we acquired the tools required to unveil the physics hidden in the formal construction introduced in Section 3. We learned how to incorporate all the constraints to isolate the physical fields and what remains to be done is to see what happens in the presence of the dislocation condensate of Section 3: the stress photons get coupled to the dislocation condensate and these will exhibit a dual

Higgs mechanism. In Section 6.1 we will derive the precise form of the coupling between the dislocation condensate order parameter field and the stress gauge fields. In Section 6.2 we will derive the effective Gauge-field action describing the nematic states.

6.1. The charges in stress-gauge theory

Let us start using the lessons of the previous sections to find out what the physical dislocation currents are and how they interact with the projected stress-gauge fields. In the duality framework, minimal coupling acquires the meaning that it just enumerates the number of ways in which order fields can become singular. As we already discussed, dislocations enter via the minimum coupling action Eq. (31),

$$\mathcal{L}_1^{\text{dual}} = \mathrm{i}B_\mu^a J_\mu^a. \tag{85}$$

The singularities can of course only occur in physical field configurations, and because the stress gauge fields B_μ^a have unphysical components, the dislocation currents J_μ^a also correspond with a redundant set: a gauge volume has to be divided out. In the previous sections we learned how to isolate the physical content in the stress-gauge sector, and gauge invariance implies that a physical source is uniquely associated with every physical stress photon,

$$S_{BJ} = \mathrm{i}B_{-1}^T {J_{-1}^T}^* + \mathrm{i}B_{+1}^T {J_{+1}^T}^* + \mathrm{i}B_{-1}^L {J_{-1}^L}^*, \tag{86}$$

where the Fourier components of the corresponding fields are assumed. As only three physical stress photons exist, no more than three components $(J_{-1}^T, J_{+1}^T, J_{-1}^L)$ of the six components of the tensor J_μ^a are physical.

At this stage, the theory still remembers the fact that dislocations carry vectorial charges. As we discussed, the 'lower index' -1 implies that the currents $J_{-1}^{L,T}$ are space-like, meaning that these refer to literal currents in this non-relativistic problem. Using Eq. (58) the L, T components can be transformed into x, y components and (J_{-1}^x, J_{-1}^y) refers to a current of dislocations with a Burgers vector having components $\sim (n_x, n_y)$.

We have now to recall the glide principle, which we discussed at length in Section 3. This is an independent constraint which has to be imposed in order to keep the theory meaningful. It amounts to a simple symmetry condition on the spatial components of the dislocation currents $J_x^y = J_y^x$. After the projection, it becomes evident why this is an independent constraint:

$$J_{-1}^L = \mathrm{i}\hat{\omega} J_{+1}^T. \tag{87}$$

The set $\{J_{-1}^T, J_{+1}^T, J_{-1}^L\}$ is still redundant and because of the glide constraint either J_{-1}^L or J_{+1}^T may be eliminated. It is convenient to eliminate J_{-1}^L in favor of J_{+1}^T. We end up with (J_{+1}^T, J_{-1}^T) as the physical currents. These are space–time filling because both polarizations ± 1 are present (i.e., they correspond with $J_\mu^T, \mu = x, y, \tau$ before the projection). It is surprising that only a single, transversal Burgers flavor T is left. It means that when the dislocations would only communicate via the long range elastic forces they would at least in 2 + 1D become indistinguishable from either vortices or just electromagnetically charged scalar matter in the static limit. A difference is that the stress gauge fields have a richer structure than electromagnetic fields, and this renders the dynamics richer. By inserting the glide condition Eq. (87) into Eq. (86) it follows that

$$S_{BJ} = \mathrm{i}B^T_{-1}{J^T_{-1}}^* + \mathrm{i}(B^T_1 - \mathrm{i}\hat{\omega}B^L_{-1}){J^T_{+1}}^*. \tag{88}$$

The coupling to the $B^T_{\pm 1}$ fields is as usual, and the novelty is that the J^T_{+1} current couples to the B^L_{-1} field with a charge $g \sim \omega$. Matters further clarify by transforming to normal coordinates, Eq. (69),

$$S_{\mathcal{B}J} = \mathrm{i}B^T_{-1}J^T_{-1} - \frac{\mathrm{i}}{\sqrt{1+\hat{\omega}^2}}\left[(1-\hat{\omega}^2)\mathcal{B}^T_1 - 2\hat{\omega}\mathcal{B}^L_{-1}\right]{J^T_{+1}}^*. \tag{89}$$

The gauge field action is given by Eq. (70). In the scaling limit, approaching the vacuum, the higher time derivatives $\sim \hat{\omega}^2$ are clearly irrelevant. The $\mathcal{B}^L_{-1}$ mode decouples, and one ends up with a structure which is except for a velocity anisotropy just like electromagnetism coupled to a scalar Higgs field.

6.2. The dual dislocation-Higgs mechanism

Let us now recall the discussion in Section 3. When the dislocation loops blow out the dislocation worldlines entangle, forming a Bose condensate described by the order parameter field Ψ. This is in turn minimally coupled to an effective gauge field $n_a B^a_\mu$. The condensate itself is living in the static $\omega \to 0$ limit and the relevant coupling becomes,

$$S_{\mathcal{B}J}(\omega \to 0) = \mathrm{i}\left(B^T_{-1}{J^T_{-1}}^* + B^T_{+1}{J^T_{+1}}^*\right) = \mathrm{i}B^T_\mu J^{T*}_\mu. \tag{90}$$

Interestingly, due to the intervention of the glide principle, the Burgers vectors are in a sense 'eaten' and turned into a scalar, purely transversal label. This has the interesting consequence that if dislocations would only interact by the long range deformations as parameterized by the stress gauge fields true nematic order would be impossible at least in the isotropic medium. As we explained in Section 3, the breaking of rotational invariance involves an ordering of the Burgers vectors and according to Eq. (90) the long range fields do not carry information regarding the vectorial nature of the topological charges. However, starting from a real crystal this information will be available at short distances and this translates into the coupling $|\Psi|^4 Q_{ab}Q_{ba}$.

Recalling the static Ginzburg–Landau–Wilson action for the dislocation disorder field theory Eq. (39) and taking into account that only the transversal components of the stress Gauge fields carry charge, it follows from Eq. (35),

$$S_{\text{disorder}} = \int \mathrm{d}^2x\left[|(\partial_a - \mathrm{i}n_T B^T_a)\Psi|^2 + m^2_\Psi|\Psi^2| + w_\Psi|\Psi^4| - r|\Psi|^4 Q_{ab}Q_{ba}\right]. \tag{91}$$

This action shows that when m^2 turns negative, a Higgs mass will be generated in a conventional way: $\Psi \to |\Psi|\exp(\mathrm{i}\phi), |\Psi| > 0$ and the covariant derivative term $\sim |\partial_a\phi - n_T B^T_a| \to |n_T B^T_a|^2$.

In addition, we also must to take into account the *dynamical* coupling to the $\mathcal{B}^L_{-1}$ photon. Although power counting demonstrates that this coupling is clearly irrelevant for the vacuum, it plays a crucial role in the dynamics. This full dynamics is easily derived. Writing Eq. (89) as

$$\mathcal{L}_{BJ} = \mathrm{i} g^a_{k,l}(\hat{\omega}) B^{a*}_k J^T_l + \text{h.c.}, \tag{92}$$

where the charges g are zero except for

$$\begin{aligned} g^T_{-1,-1} &= 1, \\ g^T_{+1,+1} &= \frac{(1-\hat{\omega}^2)}{\sqrt{1+\hat{\omega}^2}}, \\ g^L_{-1,+1} &= \frac{(-2\hat{\omega})}{\sqrt{1+\hat{\omega}^2}}. \end{aligned} \tag{93}$$

It follows directly that the Meissner term acquired by the stress gauge fields in the dual condensate has to be

$$\mathcal{L}_{\text{Meissner}} = \frac{q_0^2}{2\mu} \sum_{kk'aa'} |n_T|^2 g^a_{k,l} g^{a'}_{k',l} \left(B^{a*}_k B^{a'}_{k'} + \text{h.c.} \right). \tag{94}$$

The square root of the Higgs mass (q_0) is an inverse 'magnetic' penetration depth which we will soon learn to appreciate as a shear penetration depth. Written explicitly,

$$\mathcal{L}_{\text{Meissner}} = \frac{q_0^2 |n_T|^2}{2\mu} \left[|B^T_{-1}|^2 + 4 \frac{\hat{\omega}^2}{1+\hat{\omega}^2} |\mathcal{B}^L_{-1}|^2 - \frac{2\hat{\omega}(1-\hat{\omega}^2)}{1+\hat{\omega}^2} (\mathcal{B}^{T*}_1 \mathcal{B}^L_{-1} + \text{h.c.}) \right]. \tag{95}$$

In combination with the 'Maxwell' action Eq. (70), we arrive at a next central result, describing the effective total stress-gauge field action characterizing the nematic dual of elasticity,

$$\begin{aligned} \mathcal{L}^{\text{eff}}_{BB} = \frac{1}{4\mu} \Bigg[& (2\omega^2 + q^2 + 2|n_T|^2 q_0^2) |B^T_{-1}|^2 + \left(\omega^2 + q^2 + 8|n_T|^2 q_0^2 \frac{\hat{\omega}^2}{1+\hat{\omega}^2} \right) |\mathcal{B}^L_{-1}|^2 \\ & + \left(\frac{(1-v)\omega^2 + q^2}{1+v} + 2|n_T|^2 q_0^2 \frac{(1-\hat{\omega}^2)^2}{1+\hat{\omega}^2} \right) |\mathcal{B}^T_{+1}|^2 \\ & - 4|n_T|^2 q_0^2 \frac{\hat{\omega}(1-\hat{\omega}^2)}{1+\hat{\omega}^2} (\mathcal{B}^{T*}_1 \mathcal{B}^L_{-1} + h.c.) \Bigg]. \end{aligned} \tag{96}$$

This expression characterizes completely the nematic duals of the crystal, in the same way that the Meissner action is fully representative for the universal characteristics of the superconducting state. A remaining issue is how this action knows about the presence or absence of rotational symmetry breaking. This information enters via the factors $|n_T|^2$. Neglecting the frequency dependency of the charges for clarity, terms of the kind $|n_T B^T_\mu|^2$ can be rewritten as

$$|n_T B_\mu^T|^2 = (\hat{q}_y n_x - \hat{q}_x n_y)^2 |B_\mu^T|^2 = Q^2 \left(\frac{1}{2} + \hat{q}_y^2 Q_{xx} + \hat{q}_x^2 Q_{yy} - 2\hat{q}_x \hat{q}_y Q_{xy} \right) |B_\mu^T|^2 \qquad (97)$$

in terms of the unit directors Q_{ab} introduced in Section 3. One infers that the effect of the director order $\langle |Q_{ab}| \rangle \neq 0$ is in causing the shear-Higgs mass to become orientation dependent. The glide principle lies at the origin. The glide constraint is responsible for the coupling being entirely in the transversal channel. The Higgs mass has to do with the kinetic energy of the dislocations and, as we explained in Section 3, glide does imply that the dislocation motions are 'directed' by their Burgers vectors, which in turn order along the director. A surprising feature is that the order parameter field Ψ is still fully two (space) dimensional, and the condensate is just conventional. The rational is found in Section 3.3 and Appendix B. The dimensional reduction ('dynamical compactification') implied by Eq. (97) in the presence of director order as will be discussed in Section 8 becomes only manifest on the level of the stress-photons.

As we discussed in Section 3 at length, the other possible state of nematic quantum matter is the Coulomb nematic which is not breaking rotational invariance, such that $\langle |Q_{ab}| \rangle = 0$. This is the more basic problem which we will analyze first. In the absence of director order, it follows directly from Eq. (97) that $|n_T|^2 = Q^2/2$, a simple scalar which can be absorbed in the Higgs mass $Q^2 q_0^2/2 \rightarrow q_0^2$. Therefore, in the Coulomb nematic one is dealing with an isotropic Higgs mass and this will be further analyzed in the next section. As we will see in Section 8, the ordered nematic is just a spatially anisotropic extension of this theme.

7. The Coulomb nematic as a topological superfluid

The hard work is done and it becomes now a matter of computing observables to establish what it all means physically. The Coulomb nematic is the most basic example, and in this section we will use it to extract the most important results of this paper. We will demonstrate that the dual dislocation-Higgs condensate is at the same time a superfluid which is indistinguishable from a conventional superfluid except for the presence of the topological nematic order. The mechanism driving the superfluidity is novel. The order parameter in a conventional superfluid is governed by off-diagonal long range order in terms of the field operators associated with the constituent particles ('interstitials' in our jargon). We are dealing with a state having off-diagonal long range order in terms of the dual dislocation fields, objects corresponding with an *infinity* of constituent particles. The dislocation condensate is minimally coupled to the stress photons, and these acquire a Higgs mass. However, they do so in a special way. By imposing the glide condition, only the shear components of the stress photons acquire a mass: as we discussed in Section 3, compression is decoupled. The effect is that the dual dislocation condensate is characterized by two massive shear 'photons' and a massless purely compressional mode. According to a theorem by Feynman a bosonic state carrying an isolated massless compression mode has to be a superfluid. Hence, we have discovered a way to understand superfluidity without invoking Bose condensation of particles: a superfluid is a Bose-solid having lost its rigidity against shear stresses!

7.1. The stress photons in the Coulomb nematic state

The starting point is the action Eq. (96) and still some work has to be done. The 'magnetic' transversal phonon-like stress photon B^T_{-1} is easy: its spectrum becomes in real frequency $\omega = \sqrt{q^2 + q_0^2}$, implying that this photon just acquires a Higgs mass $\omega_0 = q_0$. One observes that its coupling constant is just $1/\mu$, the inverse of the shear modulus. This is therefore a pure shear mode and this sheds already some light on the nature of this dual Higgs condensate. The elastic medium is associated with translational symmetry breaking with the consequence that shear forces can propagate over infinite distances. Dislocations restore the translational invariance with the consequence that shear rigidity vanishes in the dual condensate. The physical interpretation of the dual Higgs mechanism is now obvious: the dislocation condensate just takes care of the requirement that shear forces can only propagate over finite distances in the liquid state, and q_0 (carrying the dimension of inverse length) just parameterizes the inverse length scale over which shear can penetrate into the fluid: the 'shear penetartion depth'.

Eq. (96) tells a more complicated story regarding the fate of the longitudinal phonon-like stress photon $\mathcal{B}^L_{-1}$ and the 'third photon' $\mathcal{B}^T_{+1}$ in the presence of the dislocation condensate. The Meissner effect causes a complicated, frequency dependent mode coupling between these two stress photons. With some effort, the action be diagonalized

$$\mathcal{L}^{\text{eff}}(\mathcal{B}) = \frac{1}{4\mu}\left[(2\omega^2 + q^2 + 2q_0^2)|B^T_{-1}|^2 + \frac{1}{(1+v)}\sum_{\pm}\left(\omega^2 + \left(1+\frac{v}{2}\right)q^2 + (1+\hat{\omega}^2)(1+v)q_0^2 \pm \left\{\left(v\left(\omega^2 + \frac{q^2}{2}\right) + (1+\hat{\omega}^2)(1+v)q_0^2\right)^2 - 2v(1+v)(1-\hat{\omega}^2)q^2q_0^2\right\}^{1/2}\right)|\mathcal{B}_{\pm}|^2\right] \tag{98}$$

and we will analyze this result in more detail in the remainder of this section.

By considering the zero-frequency limit one can already arrive at a crucial insight,

$$\lim_{\omega\to 0}\mathcal{L}^{\text{eff}} = \frac{1}{4\mu}\left[(q^2 + 2q_0^2)|B^T_{-1}|^2 + q^2|\mathcal{B}_+|^2 + \left(\frac{q^2}{1+v} + 2q_0^2\right)|\mathcal{B}_-|^2\right] \tag{99}$$

Although the penetration depths are different, both the B^T_{-1} and $\mathcal{B}_-$ stress photons have acquired a mass. However, the $\mathcal{B}_+$ photon has remained massless. The elastic medium carries both shear and compression rigidity. Different from shear, compression should be left unaffected by the dislocation condensate: as we discussed in Section 3 the theory is explicitly constructed to satisfy this requirement (the glide principle). The longitudinal phonon is of a mixed shear- and compressional nature, and so is the $\mathcal{B}^L_{-1}$ stress photon which is traveling at the same velocity. As a consequence, the longitudinal phonon is destroyed by the dislocation condensate. However, the dislocations do not communicate with compression and therefore the dual state has to be characterized by a pure massless compression mode, characterized by a velocity $c \sim \sqrt{\kappa/\rho}$. As we will see, this is precisely what is happening. Clearly, in order to turn the longitudinal phonon into a compression mode, another actor is needed and this the $\mathcal{B}^T_{+1}$ stress photon! The bottom line is that the mysterious, invisible 'third photon' of the elastic state turns into an entity with a straightforward physical interpretation in the dual state. It is not only needed to 'repair' compression, but by combining itself with the 'longitudinal' photon it gives rise to yet another physical massive 'electrical shear' mode having a different gap and velocity than the 'magnetic shear' B^T_{-1} mode. The reader might already have inferred that these are statements regarding the generic modes of the superfluid in the asymptote of strong correlations.

7.2. The long wavelength limit: isolation of compression

To derive the elastic propagator using Eq. (83) for arbitrary q, ω is quite tedious, and barely worth the effort. It reduces to the phonon propagator Eq. (84) in the regime $q^2 + \omega^2 \gg q_0^2$ and the physical interest is in first instance in the long wavelength limit $q \ll q_0$ where the dislocation condensate is in charge of the physics.

In the long wavelength limit $q \ll q_0$ matters simplify considerably. Let us focus on the 'electrical' shear- and massless compression photons. To get a handle on the $q \ll q_0$ limit, let us consider the dislocation-Meissner action in terms of the projected fields B itself, instead of the transformed $\mathcal{B}$ fields,

$$\mathcal{L}_{\text{Meissner}} = \frac{q_0^2}{2\mu} |B^T_1 - \mathrm{i}\hat{\omega} B^L_{-1}|^2. \tag{100}$$

It is directly seen that this is diagonalized by,

$$\begin{aligned} \mathcal{A}^T_{+1} &= \frac{\mathrm{i}}{\sqrt{1+\hat{\omega}^2}} \left(B^T_1 - \mathrm{i}\hat{\omega} B^L_{-1} \right), \\ \mathcal{A}^L_{-1} &= \frac{\mathrm{i}}{\sqrt{1+\hat{\omega}^2}} \left(\hat{\omega} B^T_1 + \mathrm{i} B^L_{-1} \right) \end{aligned} \tag{101}$$

such that

$$\mathcal{L}_{\text{Meissner}} = \frac{q_0^2}{2\mu} (1+\hat{\omega}^2) |\mathcal{A}^T_{+1}|^2 \tag{102}$$

showing that the $\mathcal{A}^L_{-1}$ mode decouples completely from the dislocation condensate. The problem is, however, that this transformation Eq. (101) does not remove the mode couplings associated with $v \neq 0$: $\hat{\omega}$ has reversed its sign as compared to the transformation Eq. (69). Let us see what happens when we transform Eq. (68) to the $\mathcal{A}$ basis,

$$\mathcal{L}_0 = \frac{|p|^2}{4\mu}\left[|\mathcal{A}^T_{+1}|^2 + |\mathcal{A}^L_{-1}|^2 - \frac{v}{(1+v)(1+\hat{\omega}^2)}|(1-\hat{\omega}^2)\mathcal{A}^T_{+1} + 2\hat{\omega}\mathcal{A}^L_{-1}|^2\right] \tag{103}$$

and it is immediately inferred that the prefactor of the mode coupling $\mathcal{A}^T_{+1*}\mathcal{A}^L_{-1}$ + h.c. is proportional to $\hat{\omega}(1-\hat{\omega}^2)|p|^2/(1+\hat{\omega}^2) \sim q^2$, while the mode-splitting $\sim q_0^2$, implying that in the long wavelength limit $q^2 \ll q_0^2$ this mode coupling can be neglected, and the $\mathcal{A}$ stress photons become diagonal.

The total action can be written at long-wavelength as

$$\begin{aligned}\mathcal{L}_{q\ll q_0} &= \frac{(1-v)}{4\mu(1+v)(1+\hat{\omega}^2)}\left[2\omega^2 + \frac{(1+v)}{(1-v)}q^2\right]|\mathcal{A}^L_{-1}|^2 \\ &\quad + \frac{1}{4\mu}\left[(2\omega^2 + q^2 + 2q_0^2)|B^T_{-1}|^2 + (\omega^2 + q^2 + 2q_0^2(1+\hat{\omega}^2))|\mathcal{A}^T_{+1}|^2\right]\end{aligned} \tag{104}$$

omitting terms $O(q^2/q_0^2)$. In the same limit, the propagator Eq. (83) becomes in terms of the $\mathcal{A}$ fields,

$$\begin{aligned}G_{q\ll q_0} &= -\text{const.} + \frac{(1-v)^2\omega^2}{2\mu^2(1+v)^2(1+\hat{\omega}^2)}\langle\langle \mathcal{A}^L_{-1}|\mathcal{A}^L_{-1}\rangle\rangle + \frac{2\omega^2}{4\mu^2(1+\widehat{\omega^2})} \\ &\quad \times \langle\langle \mathcal{A}^T_{+1}|\mathcal{A}^T_{+1}\rangle\rangle + \frac{2\omega^2}{4\mu^2}\langle\langle B^T_{-1}|B^T_{-1}\rangle\rangle \\ &= -\text{const.} + \frac{2}{\kappa}\frac{q^2}{\omega^2 + \frac{\kappa}{\rho}q^2} + \frac{1}{\mu}\frac{q^2 + 2q_0^2}{2\omega^2 + q^2 + 2q_0^2} + \frac{1}{\mu}\frac{3q^2/2 + 4q_0^2}{\omega^2 + 3q^2/2 + 4q_0^2}\end{aligned} \tag{105}$$

and the spectral function can be read off,

$$\begin{aligned}\mathrm{Im}[G_{q\ll q_0}] &= \frac{2q^2}{\kappa}\delta\left(\omega^2 + \frac{\kappa}{\rho}q^2\right) + \frac{q^2 + 2q_0^2}{\mu}\delta\left(2\omega^2 + q^2 + 2q_0^2\right) \\ &\quad + \frac{3q^2 + 4q_0^2}{\mu}\delta\left(\omega^2 + 3q^2/2 + 4q_0^2\right).\end{aligned} \tag{106}$$

7.3. Discussion I: The scaling limit and the continuity principle

Eq. (106) is one of the central results of this paper. It shows several highly interesting features. Most importantly, the dislocation condensate or 'dual of the crystal' is characterized by a single massless collective mode. This mode is nothing else than a true compression mode as is obvious from both the velocity and the coupling constant, set both by the compression modulus κ. The propagator is at long wavelength identical to Eq. (47) associated with a 'compression-only' solid characterized by $v = 1$. Having understood the mechanics of the dual Higgs mechanism this is not a surprise. At long wavelength compression decouples from the dislocation sources, and the compression mode is therefore not affected by the Higgs mass coming from the dislocation Bose condensate.

The fact that this state is characterized by an isolated, massless compression mode is sufficient condition to arrive at the following far-reaching conclusion:

> The dislocation condensate is at the same time a conventional superfluid.

This statement rests on a general hydrodynamic argument predating the discovery of off-diagonal long range order: it is originally due to Landau [35] and further elaborated by Feynman [36]. Landau argued that the superfluid fraction could just be characterized by a 'phonon,' or more precisely, by a propagating pure compression mode. The difference with a normal fluid (carrying also sound) is in the complete vanishing of the *shear viscosity*—although a normal fluid does not carry a reactive response towards shear forces it will show a dissipative response. From general hydrodynamic consideration it follows that the flow of a fluid with vanishing shear viscosity is *under all circumstances* a potential flow, and such a flow is: (i) dissipationless and (ii) irrotational. In any hydrodynamics textbook one finds that vorticity, dissipation and shear go hand in hand.

This hydrodynamical argument is not in conflict with ODLRO. The phase mode of the conventional interacting Bose-gas is just an isolated compression mode. The novelty of our result is that it demonstrates that Landau's hydrodynamical characterization is the more fundamental one; conventional ODLRO is just one microscopic mechanism to realize this hydrodynamics. We demonstrate an alternative mechanism. The 'crystal-dual' presented here is characterized by off-diagonal long range order in terms of the dislocations. Every dislocation can be regarded as a composite of an infinite number of constituent particles, and we took this infinity particularly serious by insisting that the constituent particles have disappeared altogether. In this sense, the overlap with the conventional Bose-gas is vanishing. The effect of the dual Higgs condensate on the flow properties of the medium are indirect. The 'true' phonons of the original crystal are propagating modes because of Goldstone protection. This Goldstone protection is not affected by the coherent dislocation Bose condensate: it inherites its dissipationless nature from the elastic medium. However, the dislocation condensate causes a Higgs mass in the shear 'sector' and an elastic medium characterized by massive shear and massless compression sustains material flows which cannot be pinned while the flows have to be irrotational. In essence, it is like the charge density wave sliding of Fröhlich. The problem with Fröhlich's superflow is that any violation of Euclidean invariance will restore dissipation: the impurity pinning. Pinning requires shear rigidity, and the sliding mode/longitudinal phonon carries shear. However, shear rigidity is vanishing in the dislocation condensate. The impurity potential is screened by the dislocations, and on scales exceeding the shear penetration depth $1/q_0$ the impurity potentials have disappeared. Hence, the dislocation condensate turns the sliding mode in a superfluid mode which cannot be pinned. At the same time this fluid is irrotational. Only at energies exceeding the shear mass gap, vortices can exist. Vorticity implies shear stresses.

Summarizing, our duality shows that the usual definition,

> a superfluid is a state of matter characterized by off-diagonal long range order in term of the field operators of the constituent bosons

should be supplemented by,

a superfluid is an elastic medium having lost its rigidity against shear stresses due to the presence of a dual dislocation condensate,

to arrive at a complete characterization. We emphasize that both viewpoints are equally correct. All one can discern in the scaling limit is the universal definition, as given by Landau:

a superfluid is a state of matter characterized by a low energy spectrum which is exhausted by a propagating, massless compression mode.

Since both 'states' are hydrodynamically indistinguishable they are actually the same state: by general principle it has to be possible to adiabatically continue the Bose-gas superfluid into our 'order' superfluid. These are just limiting 'microscopic' descriptions of the same entity. Among others, this also implies a 'don't worry' theorem regarding the problem that interstitials cannot be incorporated in the field theory. Because of continuity, it has to be that these can be 'smoothly' inserted in the theory, driving the superfluid away from the order-asymptote which is not a singular limit.

To complete the argument we still have to demonstrate that in this irrotational fluid a genuine Meissner effect should occur when electromagnetic fields are coupled in. This we reserve for Section 9.

7.4. Discussion II: the massive shear modes

At distances shorter than the shear penetration depth the differences in the description become manifest, as follows directly from Eq. (106). From the study of ^{4}Helium a clear view has developed on the nature of the excitation spectrum in a 'particle superfluid.' At higher energies, away from the scaling limit, one finds besides an incoherent continuum associated with particle-like excitations the roton-minimum. Following Feynman [36], the roton is associated with crystal-like correlations limited to length scales of order of the lattice constant (the wiggles in the structure factor). The case we consider is about a substance where the crystal-correlations extend to length scales which are still very large compared to the lattice constant. In the field theory, the lattice constant is sent to zero while the shear penetration depth is kept finite. We predict that in this regime the system is characterized by two propagating, massive modes besides the massless compression mode, see Eq. (106). The continuity argument given in the above implies that these massive modes should be meaningful, also away from the field theoretic $a \to 0$ limit. Their existence just requires that the crystalline correlations extend to a length scale (shear penetration depth) which is substantially larger than the lattice constant. There is every reason to expect that these massive shear modes are generic under these circumstances and we expect that in a bosonic substance having a substantially weaker first order crystal–superfluid transition than Helium these modes will show up. One can even wonder if the roton seen in Helium can be interpreted as a remnant of such a mode, in the case that $1/q_0$ starts to approach the lattice constant.

What is the physical nature of the massive modes? It is easily checked that both are exclusively carrying shear. The dual Higgs mechanism acts at long wavelength to completely remove the compression content from these modes, so that compression is entirely concentrated in the massless mode. It is seen from Eq. (106) that both modes have a finite pole-strength at $q \to 0$ and these would therefore show up in for instance inelastic scattering. Amusingly, deep in the disordered state the mode content in the stress-photon representation becomes visible on 'our side' of the duality. In the solid, and in the regime $q > q_0$ (next section) only two phonons are visible but when the disorder fields take over in the regime $q < q_0$, the photons take over.

It is also seen that the two shear modes are inequivalent. The B^T_{-1} stress photon carries the natural velocity $\sqrt{\mu/\rho}$ and Higgs gap q_0. The $\mathcal{A}^T_{+1}$ mode is, on the other hand, characterized by a a velocity which is larger by a factor $\sqrt{3}$ and a gap which is larger by a factor of two. How can this be, given that they both reflect the same fundamental scales (μ, ρ, $\hbar$)? Our analysis shows that it originates in the way Lorentz invariance is broken, and since this happens in a generic way in non-relativistic nature, the ratio of the gaps and the velocities can be regarded as *universal*. The analogy with electromagnetism which we introduced earlier is helpful in this regard. The B^T_{-1} photon is magnetic-like while $\mathcal{A}^T_{+1}$ is electrical-like. Since Lorentz-invariance is broken these are no longer equivalent even when $c = 1$ and this anisotropy causes the modes to be inequivalent in a universal fashion.

Summarizing, the analysis of the excitation spectrum of our topological-nematic superfluid in 2 + 1D has revealed features which should be far more general than the present case, since they are all tied to general symmetry principles. The theory is surprisingly predictive. All what is required is a bosonic substance undergoing a second order (or sufficiently weak first order) crystal–superfluid transition. By measuring the phonons on the crystal side of the transition one can establish the shear modulus μ, the mass density ρ and the Poisson ratio v. The only free parameter left on the superfluid side is the shear penetration depth $1/q_0$ and this suffices to extract the three distinct velocities and the two mass scales associated with the three modes visible on the superfluid side.

7.5. Discussion III: intermediate scales

It is quite tedious to derive the expressions for the full propagator in the regime of intermediate momenta and frequency, and barely worth the effort. One can immediately infer what is happening. Define a dimensionless gap scale,

$$\hat{q}_0 = \frac{q_0}{|p|} = \frac{q_0}{\sqrt{q^2 + \omega^2}}. \tag{107}$$

It is immediately clear that in the regime $\hat{q}_0 \ll 1$, the spectrum of the elastic state, characterized by a longitudinal- and transverse phonon, should be recovered. Hence, when $\widehat{q_0} \simeq 1$ a crossover occurs from the compression- plus two massive shear modes of the superfluid to the phonon spectrum. At this crossover, the 'electric' shear mode acquires compression character, turning into the longitudinal mode. At the same time, the 'third stress photon' is rediscovered, and one expects this third mode to gradually loose its spectral weight, to vanish completely at large momenta.

It is easy to obtain a more detailed view on these matters by considering the regime where the dimensionless gap $\widehat{q_0}$ is small but finite. Consider the diagonalized action Eq. (98). Ignoring the magnetic shear mode, this can be written as

$$\mathcal{L}^{\mathrm{eff}}_{\pm}(\mathcal{B}) = \frac{1}{4\mu(1+v)}\sum_{\pm}\Bigg(\omega^2 + \Big(1+\frac{v}{2}\Big)q^2 + (1+\hat{\omega}^2)(1+v)q_0^2 \pm |p|^2\sqrt{\Big((1+\hat{\omega}^2)^2\Big(\frac{v}{2}\Big) + (1+v)\hat{q}_0^2\Big)^2 - 2v(1+v)(1-\hat{\omega}^2)^2\hat{q}_0^2}\Bigg)|\mathcal{B}_\pm|^2. \tag{108}$$

Expanding the square root up to order $\widehat{q_0}^2$,

$$\mathcal{L}^{\mathrm{eff}}_{\pm}(\mathcal{B}) \simeq \frac{1}{4\mu}\Bigg[\Bigg(\omega^2 + q^2 + 8q_0^2\frac{\omega^2}{2\omega^2+q^2}\Bigg)|\mathcal{B}_+|^2 + \Bigg(\frac{\omega^2(1-v)+q^2}{1+v} + 2q_0^2\frac{q^4}{(2\omega^2+q^2)(\omega^2+q^2)}\Bigg)|\mathcal{B}_-|^2\Bigg]. \tag{109}$$

One recognizes the 'third photon' ($\mathcal{B}_+$) and the longitudinal phonon ($\mathcal{B}_-$). The gaps 'seen' by these stress photons are multiplied by dimensionless ratios which depend on particular combinations of spatial momenta and frequencies. However, it is immediately clear from Eq. (109) that the poles will closely approach the bare elastic stress-photon poles. To leading order in small $\hat{q}_0$ we can therefore evaluate the values of the frequency–momenta ratios multiplying q_0^2 by inserting for these ratio's their values at the positions of the bare poles. We find

$$\mathcal{L}^{\mathrm{eff}}_{\pm}(\mathcal{B}) \simeq \frac{1}{4\mu}\left(\omega^2 + q^2 + 8q_0^2\right)|\mathcal{B}_+|^2 + \frac{1-v}{4\mu(1+v)}\Bigg(\omega^2 + \frac{q^2}{1-v} + \frac{2(1-v)}{v}q_0^2\Bigg)|\mathcal{B}_-|^2. \tag{110}$$

It follows that the the third photon looses its spectral weight in a 'universal' way (i.e., independent of the Poisson ration v) in much the same way as for instance a Bogoliubov excitation in a BCS superconductor looses its weight upon exceeding the gap scale. On the other hand, the recovery of the longitudinal photon does depend on the Poisson ratio. Interestingly, one finds that for vanishing Poisson ratio the above simple expansion becomes singular. Vanishing Poisson ratio means that the compression and shear moduli become equal, $\kappa = \mu$. What is going on at this special point?

As a fortunate circumstance, the $v = 0$ case can be easily evaluated analytically. The reason is that in this case Eq. (103) is diagonal. The dual action simplifies to

$$
\begin{aligned}
S(v=0) = &\frac{\hbar}{4\mu}\int \mathrm{d}q^2\,\mathrm{d}\omega\Big[p^2|B_1^T|^2 + (p^2+\omega^2)|B_{-1}^T|^2 + p^2|B_{-1}^L|^2 + 2q_0^2(|B_{-1}^T|^2 \\
&+ |B_1^T - \mathrm{i}\hat{\omega}B_{-1}^L|^2)\Big] \\
= &\frac{\hbar}{4\mu}\int \mathrm{d}q^2\,\mathrm{d}\omega\Big((2\omega^2+q^2+2q_0^2)|B_{-1}^T|^2 + (\omega^2+q^2+2q_0^2(1+\hat{\omega}^2))|\mathcal{A}_{+1}^T|^2 \\
&+ (\omega^2+q^2)|\mathcal{A}_{-1}^L|^2\Big).
\end{aligned} \tag{111}
$$

With the exception of the B_{-1}^T mode, one cannot yet read the mode spectrum from this expression because it involves a frequency dependent gap $2q_0^2(1+\hat{\omega}^2) = 2q_0^2(2\omega^2+q^2)/(\omega^2+q^2)$. Let us consider therefore the propagator, simplifying considerably in the $v = 0$ case,

$$
G = \frac{1}{\mu} - \frac{p^2}{4\mu^2}\Big(2\hat{\omega}^2\langle\langle B_{-1}^T|B_{-1}^T\rangle\rangle + \langle\langle B_{+1}^T|B_{+1}^T\rangle\rangle + \hat{\omega}^2\langle\langle B_{-1}^L|B_{-1}^L\rangle\rangle\Big). \tag{112}
$$

Transforming to the $\mathcal{A}$s,

$$
\begin{aligned}
G = &\frac{1}{\mu}\Bigg[1 - \frac{2\omega^2}{2\omega^2+q^2+2q_0^2} \\
&+ \frac{(2\omega^2+q^2)(\omega^2+q^2)+4\omega^2q_0^2}{\Big(\omega^2+q^2+2q_0^2\Big(1+\sqrt{1+\frac{q^2}{2q_0^2}}\Big)\Big)\Big(\omega^2+q^2+2q_0^2\Big(1-\sqrt{1+\frac{q^2}{2q_0^2}}\Big)\Big)}\Bigg] \\
= &\frac{1}{\mu}\Bigg[\frac{(q_0^2+\frac{q^2}{2})}{\omega^2+q^2/2+q_0^2} + \frac{(2q_0^2+\frac{q^2}{2})}{\sqrt{1+\frac{q^2}{2q_0^2}}}\times\Bigg(\frac{\sqrt{1+\frac{q^2}{2q_0^2}}+1}{\omega^2+q^2+2q_0^2\Big(1+\sqrt{\frac{1+q^2}{2q_0^2}}\Big)} \\
&+ \frac{\sqrt{1+\frac{q^2}{2q_0^2}}-1}{\Big(\omega^2+q^2+2q_0^2\Big(1-\sqrt{1+\frac{q^2}{2q_0^2}}\Big)\Big)}\Bigg)\Bigg].
\end{aligned} \tag{113}
$$

It immediately follows that

$$
\begin{aligned}
\mathrm{Im}[G] = &\frac{1}{\mu}\Bigg[\Big(q_0^2+\frac{q^2}{2}\Big)\delta\Big(\omega^2+\frac{q^2}{2}+q_0^2\Big) + \frac{2q_0^2+\frac{q^2}{2}}{\sqrt{1+\frac{q^2}{2q_0^2}}}\Bigg(\Bigg(\sqrt{1+\frac{q^2}{2q_0^2}}+1\Bigg)\delta \\
&\times\Bigg(\omega^2+q^2+2q_0^2\Bigg(1+\sqrt{1+\frac{q^2}{2q_0^2}}\Bigg)\Bigg) + \Bigg(\sqrt{1+\frac{q^2}{2q_0^2}}-1\Bigg)\delta \\
&\times\Bigg(\omega^2+q^2+2q_0^2\Bigg(1-\sqrt{1+\frac{q^2}{2q_0^2}}\Bigg)\Bigg)\Bigg)\Bigg],
\end{aligned} \tag{114}
$$

providing a closed solution for the spectrum.

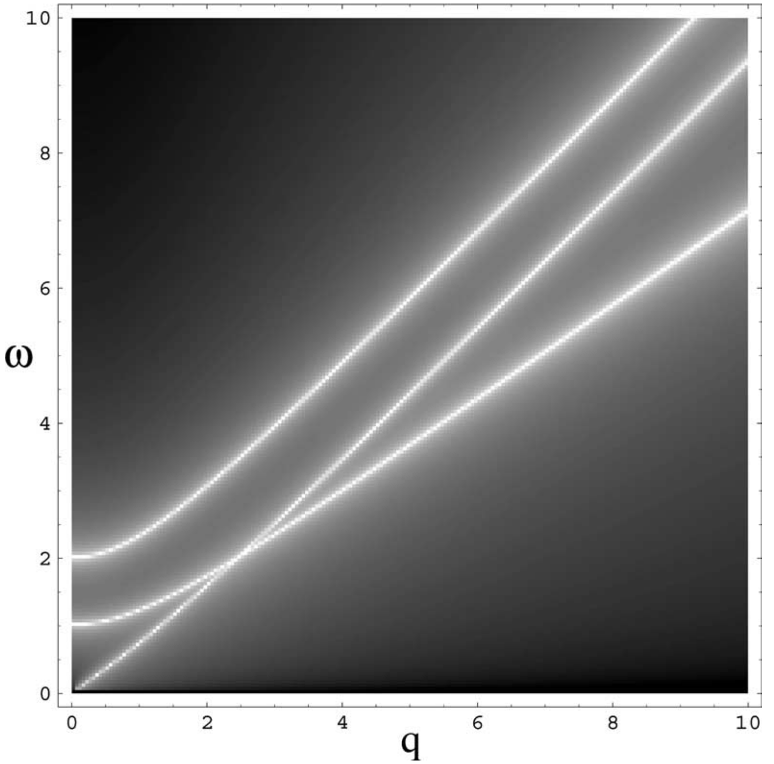

Fig. 10. The spectral function derived from the elastic propagator of the Coulomb nematic superfluid for the special case that the Poisson ratio is vanishing ($\mu = \kappa$). A small artificial broadening is added and the grey scale indicates the magnitude of the spectral function. One recognizes at small momentum q the massless compression (or superfluid phase-) mode, and the two massive shear modes. For accidental reasons, the superfluid modes do not merge with the longitudinal and transverse phonons of the solid at large momenta, and the three modes remain visible. This effect can be traced to a accidental degeneracy, see text.

In Fig. 10 the spectral function of the $v = 0$ case is shown in full, in units of $q_0 = 1$. In the long wavelength regime ($q \ll 1$) this is according to the expectations: one finds the massless compression mode and the magnetic- and electric massive shear photons with a gap ratio of two. At high frequency, $q \gg 1$, a surprise occurs: all three stress photons remain visible, while the 'longitudinal phonon' and third mode are separated by a fixed frequency $\sim q_0$, sharing the total spectral weight in the 'electrical' sector. This phenomenon has a simple explanation. In the general case the poles as function of real frequency will be determined by $\omega \sim \sqrt{q^2 + \alpha q_0^2}$ with α a number of order unity at large frequencies. For $q^2 \gg q_0^2$ this can be written as $\omega \sim q^2 + (\alpha/2)q_0^2/q$ and the mass gap will therefore not exert influences in the large frequency regime. The specialty of the $v = 0$ case is, however, that the velocity of the longitudinal- and third stress photons is the same, see Eq. (111). In other words the mode-coupling caused by the gap is at resonance, regardless the measuring frequency. Accordingly, it follows from Eq. (113) that at high frequencies the 'electrical' poles are at $\omega_\pm \sim q\sqrt{1 \pm \sqrt{2}(q_0/q) + 2(q_0/q)^2} \simeq q \pm q_0/\sqrt{2}$, split by a fixed amount regardless the frequency.

8. Breaking rotational symmetry: Is it a nematic or a smectic?

As we already announced, the ordered quantum-nematic state is a straightforward extension of the Coulomb nematic discussed in detail in the previous section. As we explained in Section 6.2, the shear Higgs mass is multiplied by a factor which depends on the orientation in (spatial) momentum space $\vec{\hat{q}}$,

$$q_0^2(\vec{\hat{q}}) = \tilde{q}_0^2\Big(1 + 2\hat{q}_y^2 Q_{xx} + 2\hat{q}_x^2 Q_{yy} - 4\hat{q}_x\hat{q}_y Q_{xy}\Big). \tag{115}$$

To gain some intuition, let us consider the case that the director is 'completely polarized.' As we explained in Section 3, $Q_{ab} = n_a n_b - 1/2\delta_{ab}$ where the vectors $\vec{n}$ are O(2) unit rotors parameterizing the Burgers vectors. 'Complete polarization' now means that the Burgers vectors are exclusively oriented in one direction. Under this condition the Qs can be parameterized as

$$\begin{aligned}
\hat{n}_x &= \cos\eta, \\
\hat{n}_y &= \sin\eta, \\
Q_{xx} &= \frac{1}{2}\cos(2\eta), \\
Q_{yy} &= -\frac{1}{2}\cos(2\eta), \\
Q_{xy} &= Q_{yx} = -\frac{1}{2}\sin(2\eta).
\end{aligned} \tag{116}$$

Let us parameterize $(\hat{q}_x, \hat{q}_y) = \big(\cos(\phi_{\mathbf{q}}), \cos(\phi_{\mathbf{q}})\big)$, as in Section 4. Eq. (115) becomes

$$q_0^2(\vec{\hat{q}}) = \tilde{q}_0^2(1 - \cos[2(\phi - \eta)]). \tag{117}$$

One observes that the Higgs mass acquires the correct π periodicity in momentum space, as required by the π periodicity of the director. Secondly, when the stress-photon has a momentum parallel to the director such that $\phi - \eta = 0 \bmod(\pi)$ the Higgs mass is vanishing while the Higgs mass is maximal when the stress-photon momentum is at a right angle relative to the director, $\phi - \eta = \pi/2 \bmod(\pi)$. This makes sense. Because of the glide principle, the dislocation 'screening' currents are running parallel to the director. The shear 'electrical-' and 'magnetic field strengths' associated with the stress photons are transversal to their propagation direction. Therefore, they acquire a maximal Higgs mass when their propagation direction is transversal to the direction of the screening currents/director. This is a most interesting result. It shows that the the spectrum of stress photons is for every orientation in momentum of the form as explained in the previous section, except that the Higgs mass has now become a function of orientation. Stronger, the Higgs mass is vanishing at the two points 'parallel' (with regard to the photon propagation direction) to the director. At these points, the shear penetration depth is diverging and the phonons of the elastic medium re-emerge.

Let us re-emphasize that this is just a consequence of the glide principle. $\langle Q_{ab} \rangle \neq 0$ means that the Burgers vectors acquire an orientational order. As we discussed, to keep compression rigidity decoupled from the dislocations one has to impose an absolute glide condition on the dynamics of the dislocations. In other words, dislocations can only delocalize in the direction of their oriented Burgers vectors. Their kinetic energy translates into the shear Higgs mass and the result is that in the directions precisely perpendicular to the director shear is decoupling from the dislocation condensate, and the phonons re-emerge.

Let us re-emphasize that is not correct to view this condensate as a quasi-1D form of matter, despite the fact that the dislocations currents are spontaneously oriented. The reasons are given in Section 3 and Appendix B. The dislocations can be created and annihilated everywhere in the 2D space and this determines the dimensionality of the effective field theory. Accordingly, the disorder field Ψ parameterizing the dislocation superfluid is just a conventional 2 + 1D GLW order parameter field. The 'dimensional' reduction resides in the way its rigidity is transferred to the stress photons. Their screening requires a real motion of the dislocations (shear screening currents) and these are guided by the director as implied by the glide constraint, causing an anisotropic shear Higgs mass.

More than anything else discussed in this paper, this 'Higgs nematic' should be considered as an entity which only carries a meaning in the field theoretic description. As we repeatedly emphasized, in real condensed matter systems there will always be a finite density of interstitial defects in the fluid states. These interstitials will directly corrupt the glide constraint; although is not clear to us how to incorporate such an interstitial gas in the present formalism it is clear that the presence of interstitials will immediately liberate climb. When climb is possible, dislocations will carry kinetic energy perpendicular to their Burgers vectors and the result has to be that the shear Higgs mass is finite in all directions. We can safely conclude that the quantum nematic derived from normal matter will be a superfluid (i.e. finite shear penetration depth in all directions) although one which might be highly anisotropic. Upon closely approaching the 'order limit' it will be a much better superfluid in the direction parallel to the director than in the perpendicular direction. We will discuss this further in the context of the superconductor, Section 9.

Another caveat is that a priori the director is a fluctuating quantity, as we discussed at length in Section 3. Considering Eqs. (115)–(117) in a naive fashion, fluctuations of the director seem to imply that a mass gap would open up everywhere at the moment that these fluctuations would become noticeable. Local fluctuations would render $\langle |Q_{ab}| \rangle < 1$ and this implies $q_0^2(\vec{\hat{q}}) > 0$ for all $\hat{q}$s. This cannot be correct, as can be inferred from simple physics considerations. As long as the director is ordered a correlation length is present. At distances shorter than this length the director is fluctuating and thereby the Burgers vector undergo directional fluctuations. Hence, at these small scales shear screening currents flow also in 'wrong' directions.

However, at length scales exceeding the correlation length director fluctuations become invisible and at these large scales only dislocation currents can flow which are precisely conforming to the director. Clearly, the fate of the phonons at the massless points is about macroscopic distances and here the director should rule in an absolute fashion. This is of course no more than a qualitative argument and this interesting problem deserves a more careful consideration, which we reserve for future study.

Despite these reservations, one can take the field theory at face value, to wonder what our finding might mean in general. This state is quite peculiar: a superfluid in all directions, except for precisely two orientations where it behaves like a solid. One could argue that it is semantically more correct to call it a 'smectic .' It should be emphasized, however, that this 'glide smectic' has nothing to do with conventional (classical) smectics and the quantum generalization thereof recently introduced by Kivelson and Lubensky et al. (also called the 'sliding phases') [20,26]. Invariantly, one starts out with special microscopic conditions such that the microscopic constituents want to form fluid- (smectic A) or algebraically ordered (smectic B, Luttinger liquids) layers. These layers then form a periodic stack, breaking the translational symmetry in the perpendicular direction. The main challenge is then to show that the layers stay fluid despite the symmetry breaking in the third direction. The rigidity of these smectics is typically governed by elastic actions of the form (3 + 1D, z is the stacking direction),

$$S \sim \int \mathrm{d}\Omega \left[B(\partial_z u)^2 + \rho(\partial_\tau u)^2 + K\left(\partial_x^2 + \partial_y^2\right)^2 + \cdots \right] \tag{118}$$

carrying only a propagating, longitudinal phonon mode in the stacking direction. One notices that this is very different from our 'glide smectic' where the full phonon response (longitudinal- and transversal modes) of the solid is recovered in the 'ordered' direction.

In conclusion, it appears that we have identified a truly novel state of quantum matter. It can only be literally realized in an unphysical limit (absence of interstitials). However, this limit can at least in principle be approached arbitrarily closely and the 'glide smectic' is potentially of relevance for the physics of the cross-over regime. Starting from the ultraviolet, one expects first to see the physics of the crystal (the phonons), to subsequently find that the phonons turn into the stress photons of the superfluid in one direction while they survive in the perpendicular direction. Only at the largest scales the latter also turn into the modes of the superfluid. From the theoretical side, the construction gives away a curious motive which makes one wonder if it might be of relevance elsewhere. In the conspiracy of spontaneous rotational symmetry breaking and the glide principle, both associated with the topological charge of the dislocation, a *dynamical compactification mechanism* has emerged. One can equally well view the above as an example of curling up an extra dimension. An observer which can only exchange information with other observers by employing shear stresses is locked up into the dimension perpendicular to the director in the static limit, to only infer about the existence of the extra dimension by investing a large energy.

9. The superconductor as the dual of the bosonic Wigner crystal

Up to this point we have focussed on elastic matter which is decoupled from other gauge fields. The outcome is that this neutral 'crystal' dualizes into the neutral superfluid. A next issue is what happens when the crystal is coupled to other gauge fields, and the gauge fields of interest is of course electromagnetism. The starting point is the electromagnetically charged crystal which one could call a charge density wave or a Wigner crystal. Traditionally, the name Wigner crystal seems reserved to the crystalline state of electrons realized at low densities in the continuum. Such a crystal carries a fermionic statistic and is thereby beyond the scope of this paper. However, adding the prefix 'bosonic' one might want to interpret it as a crystal composed of electromagnetically charged bosons, for instance the charged ordered state of Cooper pairs breaking spontaneously translational symmetry.

We asserted in the previous chapter that the dual of the neutral crystal is a superfluid. If correct, this should imply that the dual of the bosonic Wigner crystal has to be a superconductor. Symmetry does not allow for any other outcome. Upon coupling the superfluid matter fields to the electromagnetic gauge fields, the only possible outcome is the Higgs–Meissner phase.

The case can be made even more rigorous. We showed in Section 7 that the neutral case is characterized by an isolated propagating compression mode. Wen and Zee [37] presented some time ago a theorem demonstrating that such a state has to exhibit an electromagnetic Meissner effect when such a medium acquires electrical charge. Their argument is quite elementary. A compression mode can be dualized in a pair of compression 'photons' and these are coupled to the EM photons. Integrating the former results in a mass gap for the latter.

A priori, it is far from obvious that the Meissner phase will emerge in the dislocation condensate. Consistency requires that the field theory should only know about the lowest order derivatives (i.e., linear elasticity) and in this order the displacement fields only couple to electrical fields and not to magnetic fields. How can this generate Meissner? The next puzzle is that the dual of the crystal is a Bose condensate of dislocations, but it is easily demonstrated that the dislocations are not directly coupled to electromagnetism. Somehow, in order for the dual to be a superconductor, the electromagnetic screening currents have to be carried by the remnants of the elastic medium, since the dislocations themselves are incapable of carrying electromagnetic currents.

As we will show, this riddle has an amazing resolution. We will present a Higgs mechanism which is completely different from what is found in the textbooks. At the same time, the continuity arguments we used in the previous section apply equally well in this case, meaning that this interpretation is nor better and neither worse than the textbook one: it just offers an equivalent viewpoint on the nature of the superconducting state as far as the scaling limit is concerned, acquiring only a different meaning when one goes away from the scaling limit. This new mechanism can be summarized as follows:

> The Meissner effect lies in hide in the bosonic Wigner crystal, to get liberated when shear rigidity becomes short ranged.

The full theory of the electrodynamics of the Wigner crystal and its dual 'order superconductor' is rather tedious. However, using our helical projections the problem factorizes in a magnetic sector where the Meissner effect resides and an electrical sector responsible for the physics of the plasmons. The magnetic sector is quite simple technically and will be discussed in full here. The electrical sector involves some tedious algebra and will be discussed in a future publication.

9.1. Basics: electrodynamics and linear elasticity

Let us first review the standard derivation of the coupling between the EM field and the charged crystal. We start with the atomistic view on the elastic medium as a composite of charged particles. Every individual (non-relativistic) particle positioned at $\vec{R} = (R_x, R_y)$ carries a charge e and it interacts with the field according to,

$$S_{\mathrm{EM}} = e \int \mathrm{d}\tau \left(A_\tau(\vec{R}) - \frac{1}{c} A_a(\vec{R}) \partial_\tau R_a \right). \qquad (119)$$

For convenience, we use units of time measured in units of length with a conversion factor given by the phonon velocity $c_{\mathrm{ph}}^2 = 2\mu/\rho$. The velocity of light c in Eq. (119) thereby corresponds with the ratio of the light velocity to the phonon velocity, which will turn into the true light velocity when the phonon velocity is reinserted in the final outcomes.

Consistency with linearized quantum-elasticity requires that only the leading order in the gradient expansion for the phonon–photon coupling should be kept. Writing $\vec{R} = \vec{R}_0 + \vec{u}$ and expanding to lowest order,

$$S_{\mathrm{EM}} = \int \mathrm{d}^2 x \, \mathrm{d}\tau \left[(n_e e) u^a \partial_a A_\tau - \left(\frac{n_e e}{c} \right) A_a \partial_\tau u^a + \frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} \right], \qquad (120)$$

where we explicitly included the Maxwell term of the electromagnetic fields. The above amounts to the simple statement that for infinitesimal displacements the electrical fields exert a force on the displacements. The coupling to magnetic fields emerge only in the next order of the gradient expansion and these torque stresses belong to the realms of second gradient elasticity, to be omitted in the limit that the lattice constant is sent to zero.

Let us now reconsider the transformation to stress representation, keeping track of the coupling to electromagnetic fields. Including Eq. (120) in the duality transformation presented in Section 3.2, we find after the H–S transformation, and reshuffling of derivatives,

$$S = \int \mathrm{d}\Omega \left[-\frac{1}{2} \sigma^a_\mu C^{-1}_{\mu\nu ab} \sigma^b_\nu + \sigma^a_\mu \partial_\mu u^a_P + [u^a_P \partial_a A_\tau - A_a \partial_\tau u^a_P] \right.$$
$$\left. + u^a \partial_\mu (-\sigma^a_\mu + A_\tau \delta_{\mu,a} + A_a \delta_{\mu,\tau}) + \frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} \right]. \qquad (121)$$

By integrating over the smooth configurations u^a it follows that all components of the quantity,

$$\tilde{\sigma}^a_\mu \equiv \sigma^a_\mu - A_\tau \delta_{\mu,a} - A_a \delta_{\mu,\tau}. \qquad (122)$$

are separately conserved,

$$\partial_\mu \tilde{\sigma}^a_\mu = 0. \tag{123}$$

The fields σ^a_μ are again the components of the stress tensor, now satisfying the equation of motion $-\partial_\mu \sigma^a_\mu = f_a$ where f_a is the ath component of the net applied external force density, here the electrical field density. In the present dynamical context this just corresponds with the equations of motions in the presence of an electrical field. The electrical fields $\vec{E}$ are

$$E_a = -\frac{1}{c}\partial_\tau A_a - \partial_a A_\tau \tag{124}$$

and using the stress–strain relations Eq. (26) for the isotropic medium one finds explicitly the familiar real time equations of motions for the displacement fields,

$$\begin{aligned}\rho\partial_t^2 u_x &= \Big((\kappa+\mu)\partial_x^2 + \mu\partial_y^2\Big)u_x + \kappa\partial_x\partial_y u_y + n_e e E_x,\\ \rho\partial_t^2 u_y &= \kappa\partial_x\partial_y u_x + \Big(\mu\partial_x^2 + (\kappa+\mu)\partial_y^2\Big)u_y + n_e e E_y.\end{aligned} \tag{125}$$

9.2. Photons and stress photons

Up to this point it is of course standard. In the mean time, the reader should have recognized that the stress gauge fields, Kleinert's invention, have a remarkable capacity to generate deep insights. As we perceive it, the remarkable powers of the concept become fully visible when one in interested in the faith of the electromagnetic photon in the dual condensate.

It starts out with the simple observation that since the full stress tensor $\tilde{\sigma}$ (including the EM forces) is divergenceless,

$$\tilde{\sigma}^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda. \tag{126}$$

The stress gauge fields therefore 'include' the electromagnetic forces. The elastic components of the stress fields can therefore be written as

$$\sigma^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda + A_\tau\delta_{\mu,a} + A_a\delta_{\mu,\tau}. \tag{127}$$

The local conservation law on the total stress can be imposed on the action Eq. (121) by inserting this identity. After some algebra,

$$\begin{aligned}S = \int \mathrm{d}\Omega\bigg[&-\frac{1}{2}\left(\epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda + A_\tau\delta_{\mu,a} + A_a\delta_{\mu,\tau}\right)C^{-1}_{\mu\mu' ab}\Big(\epsilon_{\mu'\nu'\lambda'}\partial^{\nu'}B^b_{\lambda'} + A_\tau\delta_{\mu',b} + A_b\delta_{\mu',\tau}\Big)\\ &+ \mathrm{i}B^a_\mu J^a_\mu + \frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\bigg].\end{aligned} \tag{128}$$

One finds that the dislocation currents J^a_μ are sources for just the 'total stress' gauge fields B. Introducing the quantity ('elasticity dressed' electromagnetic fields),

$$\Phi^a_\lambda = (n_e e)C^{-1}_{\mu\mu' ab}\epsilon_{\mu\nu\lambda}\partial_\nu\left(A_\tau\delta_{\mu',b} + \frac{1}{c}A_b\delta_{\mu',\tau}\right) \tag{129}$$

we find that the action can be rewritten in the simple form,

$$S = \int \mathrm{d}^2x\,\mathrm{d}\tau \bigg[-\frac{1}{2}\left(\epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda\right) C^{-1}_{\mu\mu' ab}\left(\epsilon_{\mu'\nu'\lambda'}\partial_{\nu'} B^b_{\lambda'}\right) - B^a_\mu\left(\mathrm{i}J^a_\mu + \Phi^a_\mu\right) + \frac{n_e e^2}{2}\left(A_\tau\delta_{\mu,a} + \frac{1}{c}A_a\delta_{\mu,\tau}\right) C^{-1}_{\mu\mu' ab}\left(A_\tau\delta_{\mu',b} + \frac{1}{c}A_b\delta_{\mu',\tau}\right) + \frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\bigg]. \quad (130)$$

The first- and last term are the now familiar stress- and electromagnetism Maxwell terms. The second term shows that the dressed EM fields Φ couple minimally to the stress-gauge fields. It is also seen that the EM fields do not couple *directly* to the dislocations: terms of the form AJ have cancelled out and in this sense are dislocations electrically neutral. However, both the dislocations and the EM potentials couple to the stress gauge fields. Although the direct coupling is absent, dislocations do interact with EM fields, albeit in an indirect way by the influences they both exert on the intermediary elastic medium.

The real surprise is in the third term. By just rewriting the action in terms of the stress gauge fields, a Meissner term has emerged! Consider the space-like EM potentials occurring in the third term. Because of the delta functions, only the time-like components of the inverse elastic tensor matter and $C^{-1}_{\tau\tau ab} = \delta_{ab}/\rho$. In other words, this action contains the term $(n_e^2 e^2/\rho c^2)(A_x^2 + A_y^2)$. The prefactor carries the dimension of inverse area,

$$\frac{1}{\lambda_L^2} = \frac{4\pi n_e^2 e^2}{\rho c^2} \quad (131)$$

and λ_L is recognized as the London penetration depth, expressed in the relevant dimensionful quantities characterizing this problem. This is remarkable. Just by representing the problem of the charged elastic medium in terms of stress gauge fields, the electromagnetic sector acquires automatically a Meissner term. In the particle language one has to work hard to discover the Meissner as a ramification of the off-diagonal long range order. In this order language, when one knows to use the natural representation (stress gauge fields) Meissner is around all along, and the problem is actually to get rid of it! In the absence of dislocations $J^a_\mu = 0$ it seems to be still present. However, this corresponds with the elastic/crystalline state and the crystal is surely not a superconductor; the Meissner term has to vanish from the effective action describing the electrodynamics of the crystal. As we will show, this is indeed the case. The EM photons couple linearly to the stress photons via the term $B\Phi$. The stress photons have to be integrated to arrive at the effective action for the electrodynamics. As we will show, these integrations produces a counter term which is precisely compensating the Meissner term in the bare action. However, when the shear photons acquire their Higgs mass due to the presence of the dislocation condensate, this compensation is no longer complete and a true Meissner term remains demonstrating that the dual state is a superconductor. The reader can now appreciate the meaning of the metaphor we presented earlier: in the crystal, the Meissner effect is around (Eq. (130)), but it is lying in hide because it is 'eaten by the phonons,' to get liberated in the dual fluid, because of the mass gap in the stress-photon sector associated with the finite range of shear.

9.3. The effective electromagnetic action

The expressions Eqs. (129) and (130) appear at first sight as quite complicated, although they represent no more than the coupling of phonons and photons in the Wigner crystal (ignoring the dislocations). Matters simplify considerably by using the helical projections introduced in Section 4. To establish the connections with the familiar context of electromagnetism, it is easily shown that the magnetic- (B^2) and electrical (E^2) energy densities in momentum space are simply given by,

$$\begin{aligned} B^2(\vec{q},\omega) &\sim q^2|A^{-1}|^2, \\ E^2(\vec{q},\omega) &\sim (\omega/c)^2|A^{-1}|^2 + ((\omega/c)^2 + q^2)|A^{+1}|^2 \end{aligned} \tag{132}$$

and the electromagnetic Maxwell action becomes

$$\begin{aligned} S_{\text{Maxwell}} &= \frac{1}{16\pi c}\int \mathrm{d}\Omega F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{16\pi c}\int \frac{\mathrm{d}^2q\,\mathrm{d}\omega}{(2\pi)^3}(\omega^2 + c^2q^2)\left(|A^{+1}|^2 + |A^{-1}|^2\right). \end{aligned} \tag{133}$$

The total action for the Wigner crystal becomes

$$S_{\text{tot}} = S_{\text{Maxwell}} + S_{\text{stress}} + S_{\text{Meissner}} + S_{\text{int}}, \tag{134}$$

where S_{stress} is the 'Maxwell' action for the stress gauge fields, Eq. (70), to be augmented with the dislocation Higgs mass, Eq. (96) when we want to address the superconductor. We still have to derive the 'pseudo-Meissner' action S_{Meissner} and the coupling between the stress- and the EM gauge fields S_{int} in terms of the projected fields.The pseudo-Meissner term reads

$$\begin{aligned} \mathcal{L}_{\text{Meissner}} &= n_e^2e^2(C^{-1})_{\mu\nu ab}(A_\tau\delta_{a,\mu} - (1/c)A_a\delta_{\tau,\mu})(A_\tau\delta_{b,\nu} - (1/c)A_b\delta_{\tau,\nu}) \\ &= n_e^2e^2\left[\frac{1}{\rho c^2}|A^{-1}|^2 + \left(\frac{(1-\hat{\omega}^2)}{\kappa} + \frac{\hat{\omega}^2}{\rho c^2}\right)|A^1|^2\right]. \end{aligned} \tag{135}$$

In the non-relativistic limit $q \gg \omega/c$, we recognize that the mass in the electrical sector $\sim |A^1|^2$ with the electrical screening length $q_p^2 = n_e^2e^2/\kappa$, which is set by the compression modulus and will translate in a plasmon-frequency via $\omega_p^2 = n_e e^2/m = c_\kappa^2 q_p^2$ with the compression velocity $c_\kappa = \sqrt{\kappa/\rho}$. As we already discussed, the surprising feature is the term $|A^{-1}|^2/\lambda_L^2$, showing that the magnetic vector potential acquires a genuine Higgs mass.

The stress- and electromagnetic gauge potentials are linearly coupled and in order to arrive at meaningful statements regarding the electrodynamics one has to explicitly integrate out the stress fields. These couplings read in terms of the projected fields,

$$S_{\text{int}} = n_e eq\left(\frac{-1}{\rho c}B^T_{-1}A^*_{-1} - \hat{\omega}\left(\frac{1}{\rho c} - \frac{1}{2\kappa}\right)B^L_{-1}A^*_1 - \frac{\mathrm{i}}{2\kappa}B^T_1A^*_1\right) \tag{136}$$

and together with S_{stress} the B fields can be integrated out. Although these integrations are in principle straightforward, the problem is in practice quite complicated, especially so in the presence of the dislocation Higgs mass, and we leave the full analysis to a future publication. However, these problems all reside in the electrical sector (i.e., the plasmons), and as a pleasant circumstance the magnetic sector is simple. We observe that in all pieces of the total action Eq. (134) the problem factorizes in a 'magnetic' part in terms of B_T^{-1} and A^{-1} and an electrical part in terms of the other helical components. Collecting all the magnetic pieces,

$$\begin{aligned} S_{\text{tot}} &= S_{\text{elec}} + S_{\text{mag}}, \\ S_{\text{magn}} &= \int \frac{d^2q\, d\omega}{(2\pi)^3} \left[\frac{1}{16\pi c^2} (\omega^2 + c^2 q^2 + c^2/\lambda_L^2) |A^{-1}|^2 - \frac{n_e e q}{\rho c} B_{-1}^T A_{-1}^* \right. \\ &\quad \left. + \frac{1}{4\mu} (2\omega^2 + q^2 + 2q_0^2) |B_{-1}^T|^2 \right]. \end{aligned} \tag{137}$$

The B fields have to be integrated to arrive at the electrodynamics of the crystal or the dual fluid. The integration rule is as usual for complex variables (Fourier components),

$$\int d\text{Re}\{l\}\, d\text{Im}\{l\} e^{-\beta \mathbf{l}^2 + i(\mathbf{l}\mathbf{a}^* + \text{c.c.})} = e^{-|\mathbf{a}|^2/\beta}, \tag{138}$$

where the standard unimportant multiplicative has been omitted. Setting β to be the propagator of the B field, $a = i(n_e e q) A^{-1}/\rho c$, and recalling that we use the velocity convention $c_{\text{ph}}^2 = 2\mu/\rho \equiv 1$, we find for the magnetic piece of the effective electromagnetic action,

$$S_{\text{eff,magn.}} = \left[\frac{n_e^2 e^2}{\rho c^2} \left(1 - \frac{q^2}{2\omega^2 + q^2 + 2q_0^2} \right) + \frac{1}{16\pi} \left(\frac{\omega^2}{c^2} + q^2 \right) \right] |A^{-1}|^2. \tag{139}$$

9.4. The Meissner effect and the screening current oscillations

Eq. (139) is our second central result because it demonstrates that the dislocation Bose condensate is at the same time a conventional electromagnetic Meissner state: it offers a complementary way to understand the phenomenon of superconductivity.

Magnetic flux expulsion is associated with the vacuum and one should therefore consider the meaning of the action Eq. (139) in the limit $\omega \to 0$. Define the shear penetration depth λ_S as,

$$\lambda_S^2 \equiv \frac{1}{2q_0^2} \tag{140}$$

Recalling the definition Eq.(131) for the London penetration depth λ_L

$$S_{\text{Magn},\omega=0} = \left[\frac{1}{\lambda_L^2} \frac{\frac{1}{(q\lambda_s)^2}}{1 + \frac{1}{(q\lambda_s)^2}} + q^2 \right] \frac{|A^{-1}|^2}{16\pi} \tag{141}$$

and the propagator for static magnetic fields follows,

$$G_{\text{magn}}(\omega = 0) = 1 \bigg/ \left\{ q^2 + \frac{1}{\lambda_L^2} \frac{\frac{1}{(q\lambda_s)^2}}{1 + \frac{1}{(q\lambda_s)^2}} \right\}. \tag{142}$$

First consider the Wigner crystal, characterized by the shear penetration depth being infinite: $\lambda_S \to \infty$. It follows directly from Eq. (142) that the Meissner term cancels. We recover the expected property of crystalline matter that it does not couple to static magnetic fields.

However, the above reveals a deep insight which we believe is far more general than the specific case we have analyzed. We perceive it as a sharpening of Feynman's ideas we referred to earlier. Let us formulate it as an explicit conjecture:

> It is sufficient condition for the Meissner phase to occur that a bosonic and electromagnetically charged elastic medium acquires a finite penetration length for the mediation of shear stresses.

We cannot prove this conjecture for the general case. However, it is obvious for the present case. At length scales large compared to the shear penetration depth, $q\lambda_S \ll 0$ the propagator becomes $1/(q^2 + 1/\lambda_L^2)$, demonstrating that in real space magnetic fields decay-like $\exp(-r/\lambda_L)$, corresponding with a conventional Meissner effect. However, at distances small compared to the shear penetration depth, $q\lambda_S \gg 0$, the medium rediscovers its heritage as a solid and the magnetic field can freely propagate, $G = 1/q^2$.

There is no doubt that the dislocation condensate is a conventional superconductor and let us once again emphasize the great differences between this Higgs mechanism and the conventional understanding based on the gas limit. Starting out from the Bose-gas, the central wheel is the macroscopic quantum entanglement of the boson worldlines, translating into off-diagonal order in terms of the boson field operators, which in turn by minimal coupling causes the vector potentials to become pure gauge. Starting out from the crystal, just by re-parameterizing the dynamics in terms of stress photons a Meissner term is generated. When shear is massless as in the crystal, the shear photons eat the Meissner. However, when the shear photons acquire a mass, necessarily involving a condensate of dislocations being the unique agents destroying translational invariance and thereby shear rigidity, shear photons are eaten themselves at long distances with the effect that the hidden Meissner turns into a physical Meissner, giving the electromagnetic its mass. It seems, all what is required for this mechanism is the existence of a stress photon representation, which is general, and the fact that their shear content acquires a mass in the fluid, which is also general. This gives reason to believe that the above conjecture might well be a theorem.

The differences between the conventional and our 'order' superconductor should become more manifest when one measures shorter distances and -times. The key difference is in the shear-length, an additional scale which is central in the 'order' description. The gas limit can be viewed as the special case of the order superconductor where the shear length has shrunk to the lattice constant. One notices that shear is never mentioned in the textbook treatments of the quantum fluids. The present theory can be viewed in this regard as an extension because we can address the question if the presence of this additional length scale causes new phenomena.

One expects something to happen when the shear length becomes larger than the London length. In practice, the London length is quite large (of order of microns) while it is natural to expect that deep inside the superfluid/conductor the shear length will be of order of the lattice constant. In order to enter the regime where the shear length becomes the largest length scale it appears as a requirement that the superfluid/conductor undergoes a second order transition into the crystalline state. This condition can in principle be satisfied when the superfluid/conductor carries a nematic order, as we discussed in previous sections. Since the shear length is diverging at this transition, by approaching it sufficiently closely one will always enter a regime where the shear length is the largest length scale.

In this regime, a new effect is found which does not seem have an analogy elsewhere in physics. Let us consider the propagator Eq. (142) in real space. The inverse of the propagator Eq.(142) is,

$$G^{-1} = q^2 + \frac{1}{\lambda_L^2(1 + \lambda_S^2 q^2)} \tag{143}$$

in case of the 'normal' superconductor, characterized by $\lambda_L > 2\lambda_S$, the poles lie on the imaginary axis at,

$$q_\pm = \pm \mathrm{im}_\pm, \qquad m_\pm = \frac{\lambda_L^2 \pm \sqrt{\lambda_L^4 - 4\lambda_L^2\lambda_s^2}}{2(\lambda_L\lambda_s)^2}. \tag{144}$$

Closing the contour in the upper half plane in the Fourier transform to real space, these lead to the standard exponential decay of the real space correlator $G(r) \sim \mathrm{e}^{-r/\lambda_L}$.

For the interesting regime where the shear length exceeds the London penetration depth ($\lambda_L < 2\lambda_S$), the situation changes drastically. At $\lambda_L = 2\lambda_S$ the four poles merge in pairs and then bifurcate once again- this time along a circle. The point $\lambda_L = 2\lambda_S$ signifies the disorder line. For $\lambda_L < 2\lambda_S$, the four poles lie on a circle q in the complex plane with radius R and phase θ,

$$q = \pm \mathrm{Re}^{\pm \mathrm{i}\theta}, \qquad R = \frac{1}{\sqrt{\lambda_L\lambda_S}}, \qquad \theta = \frac{1}{2}\cos^{-1}\left(-\frac{\lambda_L}{2\lambda_S}\right). \tag{145}$$

Hence, the poles are no longer on the imaginary axis, and one anticipates an oscillatory behavior of the real space propagators. To gain insight, let us first perform the Fourier transformation in one space dimension. The real space propagator is given by the exact expression,

$$G(r) = -\pi\sqrt{\frac{\lambda_L}{\lambda_S}}\exp\left[-r\sqrt{\frac{1+\frac{\lambda_L}{2\lambda_S}}{2\lambda_S\lambda_L}}\right]\frac{1}{\sqrt{1-\frac{\lambda_L^2}{4\lambda_S^2}}}\left[(\lambda_S+\lambda_L)\sqrt{\frac{1-\frac{\lambda_L}{2\lambda_S}}{2\lambda_S\lambda_L}}\right.$$
$$\left.\times\sin\left(r\sqrt{\frac{1-(\lambda_L/(2\lambda_S))}{2\lambda_L\lambda_S}}\right)+(\lambda_S-\lambda_L)\sqrt{\frac{1+\frac{\lambda_L}{2\lambda_S}}{2\lambda_S\lambda_L}}\cos\left(r\sqrt{\frac{1-(\lambda_L/(2\lambda_S))}{2\lambda_L\lambda_S}}\right)\right]. \tag{146}$$

This simplifies in the case that the shear length is much larger than the penetration depth ($\lambda_S \gg \lambda_L$) to,

$$G(r) = -\pi\exp\left\{-\frac{r}{\sqrt{2\lambda_S\lambda_L}}\right\}\sin\left(\frac{r}{\sqrt{2\lambda_S\lambda_L}}+\frac{\pi}{4}\right). \tag{147}$$

This is a remarkable result. It shows that that the true magnetic penetration depth corresponds with the geometrical mean of the shear- and the London length, $\lambda_M = \sqrt{2\lambda_S\lambda_L}$. Even more surprising, on the scale of this penetration depth, the electromagnetic screening currents have an *oscillatory* nature. Upon entering the superconductors, one encounters first a layer of screening currents, than weaker anti-screening currents, etcetera.

This is of course not a pathology of the 1D case—it is just a generic consequence of the location of the poles of the propagator in the complex plane. In two space dimensions the real space propagator reads,

$$G(r) = \frac{1}{2\pi(\lambda_L\lambda_S)^2}\left[a_+K_0(m_+r)+a_-K_0(m_-r)\right] \tag{148}$$

with

$$\begin{aligned} m_\pm^2 &\equiv \frac{\lambda_L^2 \pm \mathrm{i}\sqrt{4\lambda_L^2\lambda_S^2-\lambda_L^4}}{2(\lambda_L\lambda_s)^2}, \\ a_- &\equiv \frac{\lambda_L^2 - m_-^2(\lambda_L\lambda_S)^2}{m_+^2-m_-^2}, \\ a_+ &\equiv \frac{\lambda_L^2 - m_+^2(\lambda_L\lambda_S)^2}{m_-^2-m_+^2} \end{aligned} \tag{149}$$

with the Bessel function

$$K_0(x) = \int_0^\infty \mathrm{d}t\,\frac{\cos xt}{\sqrt{1+t^2}} \tag{150}$$

arising from the integral

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{\exp[\mathrm{i}\vec{k}\cdot\vec{r}]}{k^2+m^2} = \frac{1}{2\pi}K_0(mr). \tag{151}$$

In Figs. 11 and 12 we have plotted some representative cases and it is seen that the 2D propagator behaves in the same way as in one dimensions. In Fig. 11 we consider the case that $\lambda_S = 100\lambda_L$, deep in the interesting regime where the shear penetration depth is much larger than the London length. The thick line shows the behavior of the magnetic propagator Eq. (148) directly. Since the scale setting the exponential envelope is the same as the one setting the oscillation period $\sim \sqrt{2\lambda_L\lambda_S}$ the actual magnitude of the anti-screening currents is quite small. With some effort one can just recognize 1.5 full oscillations. We therefore also show the outcome after dividing out the exponential envelope $\sim \exp(-r/\sqrt{2\lambda_L\lambda_S})$ (thin line) and this shows very clearly the oscillations. In Fig. 12, we follow the same strategy, now showing how the oscillations develop as function of increasing λ_S/λ_L. In accordance with the 1 + 1D result, we find that the oscillations show up only when $\lambda_S > \lambda_L$, while their magnitude is increasing when λ_S grows. In addition, one can clearly see that the period of the oscillations $\sim \sqrt{2\lambda_L\lambda_S}$.

What is the physical implication of these findings? In the regime where the oscillations occur, the magnetic screening has to take place at lengths smaller than the shear penetration depth. In the beginning of this section we stated that at these scales the medium is rediscovering that it is actually more like a solid and that magnetic fields should freely penetrate. On closer inspection, however, the answer turns out to be more subtle. In the presence of the dislocation condensate, the solid is actually not quite recovered. At distances where the medium can carry shear stresses, it has

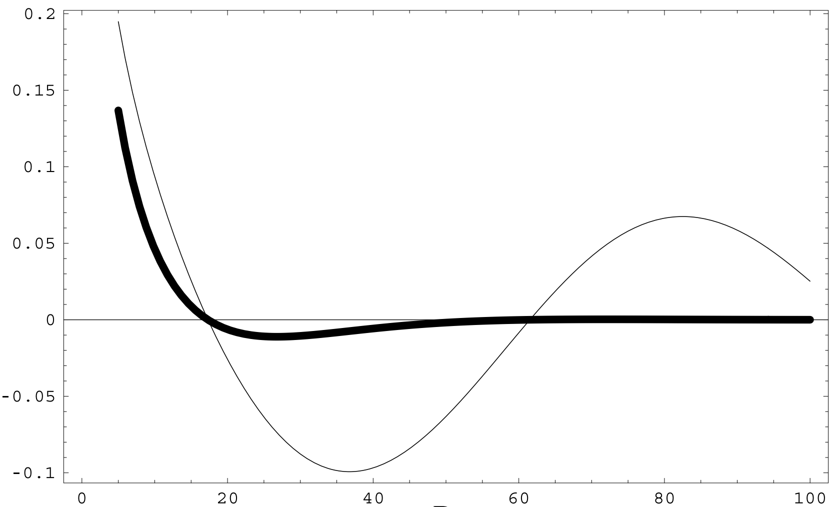

Fig. 11. The behavior of the static magnetic propagator in two space dimensions, indicating how magnetic fields vanish upon entering the ‘order superconductor.’ The unit of length is set by the London penetration depth $\lambda_L = 1$ and we consider here the case than the shear penetration depth is large, $\lambda_S = 100$, corresponding with the situation that the magnetic fields are screened by matter which is rediscovering its solid nature. The thick line corresponds with the propagator and to amplify the screening oscillations we have divided out the exponential factor $\exp(\sqrt{2\lambda_L\lambda_S})$ (thin line). The real magnetic penetration depth is $\sqrt{2\lambda_L\lambda_S}$ and one notices that on the same scale the response oscillates between diamagnetic- and paramagnetic behaviors.

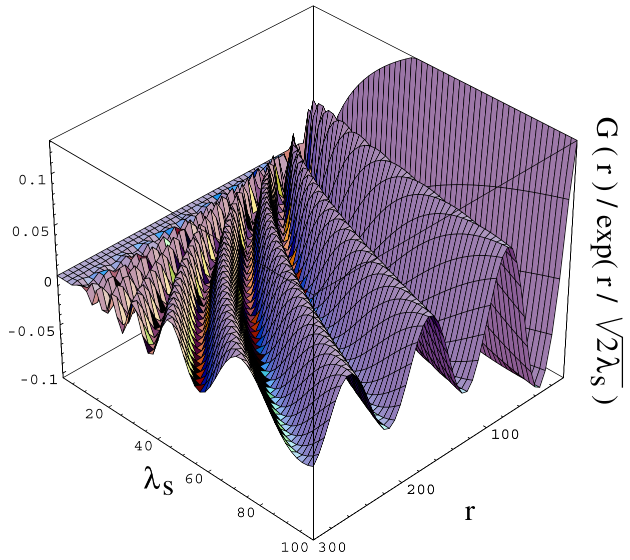

Fig. 12. Same as the thin line in Fig. 11 except that we now show how the oscillations develop as function of the shear length λ_S, again using $\lambda_L = 1$ as unit of length.

still sufficient fluid character in order to build up the currents needed to screen the magnetic field. It is doing so less efficiently than in the true fluid regime characterized by the London penetration depth. The magnetic penetration length is now given by the geometrical mean $\sqrt{\lambda_S \lambda_L}$ which is large compared to the London length, although still small compared to λ_S itself. The magnetic fields can be completely screened by the 'shearing' fluid.

The most surprising feature are the current oscillations. Is there any further physical interpretation available? At first sight one might be tempted to draw analogies with Friedel oscillations, to find out on closer inspection that there is little relationship. We have failed to find any other analogy for this phenomenon in terms of the physics which is known to us. We therefore consider it as an irreducible mathematical fact and thereby as the central prediction of this paper, given its counterintuitive nature.

9.5. Magnetic screening in the ordered nematic

Up to this point we have focussed on the simpler case of the Coulomb nematic. However, it is easily deduced what happens with the magnetic screening in the ordered nematic state. Although unphysical, it is entertaining to consider what happens in the 'glide smectic' of Section 8. We learned that the shear penetration depth is actually diverging in the direction perpendicular to the director. This in turn implies that in the direction parallel to the director the magnetic penetration depth is just the London length while the magnetic penetration depth becomes infinite in the perpendicular direction.

Considering the type II state, this has the amusing consequence that vortices actually turn into lines or 'stripes'. This sounds far-fetched but a very similar phenomenology is expected in the Luttinger-liquid smectics discovered by Kivelson et al. [26]. It is just expressing the fact that the effectively 1D screening currents are not able to screen the magnetic fields.

More realistically, one would expect that because of the liberation of climb the shear penetration depth might be very anisotropic but still finite in the climb direction. This in turn implies that the magnetic penetration depth is strongly anisotropic, and the vortices in the type II state should have a strongly elongated form. At least in the 123 high T_c superconductors a sizable anisotropy of this kind has been observed. Still, these anisotropies are quite moderate implying that if nematic order of this kind is present the system has to be relatively far removed from the order limit.

10. Conclusions

Nearing the end of this exposition, we are left with the feeling that this paper should have been written some fifty years ago in the era that the fundamental theory of superconductivity was developed. Although our initial aim was to shed light on the nature of nematic quantum fluids, we were forced in the process to reconsider the fundamentals of the theory of bosonic quantum fluids. We discovered a way to understand these fluids which is strikingly different, but complementary to the theory found in the textbooks. The superconductor and the superfluid can be understood as duals of the crystalline state, the state realized when a dislocation condensate has destroyed the translational symmetry breaking governing the elastic state.

In order to keep the mathematics under control, we could only develop the theory in full in a limiting case. This limit can be viewed as the limiting case of 'fluctuating order' (or strong correlations) in a bosonic fluid, and is defined through the requirement that interstitial degrees of freedom do not exist. This condition can never be satisfied in fluids composed of atomistic matter, and the theory is thereby not literally applicable to condensed matter systems. However, knowledge of limits is always useful. Stronger, as we argued repeatedly, since our 'order' superconductor can be adiabatically continued to the Bose-gas limit (of course, modulo nematic 'obstructions') the presence of interstitial (Bose-gas-like) excitations can only change matters quantitatively and not qualitatively. In sufficiently strongly correlated Bosonic fluids, the physics discussed in this paper can be closely approached in a finite energy crossover regime, to turn into a truly conventional superconductor only at the lowest energies.

A second main theme has been the development of a topological quantum theory addressing zero temperature nematic order. With regard to the 'order superconductivity' theme nematicity has taken the role of auxiliary device, simplifying the theory. We believe it is in this regard not essential. We did not address explicitly the transition from the nematic- into the isotropic quantum fluid driven by the condensation of disclinations. It remains to be proven that this isotropic state is in the scaling limit indistinguishable form a conventional superconductor/superconductor. However, there is every reason to believe that this is the case. Relative to the stress photons, the disclinations live 'two gradients higher' (Kleinert's double curl gauge field construction). These cannot couple to the massless compression mode of the superfluid, and they will not alter the electromagnetic Higgs mechanism presented in Section 9. To the best of our present understanding, the disclination condensate will just destroy nematicity, be it of a topological- or fully ordered kind, in the way it is envisaged in the Lammert–Toner–Rokshar gauge theory [3].

Nevertheless, we perceive our findings regarding the nature of nematic orders as quite interesting. A main result is the de-mystification of the topologically ordered nematic of Lammert et al. [3]. Although quite mysterious starting from the conventional 'lengthy molecule gas' perspective, this Coulomb nematic appears as a natural possibility in the field-theoretic framework. We emphasize that our arguments are in this regard of equal relevance in the 3D classical melting context. Topological ordered states of this kind are quite elusive. Experimentalists can only infer their existence through the presence of second order phase transitions taking place in the absence of manifest order parameters. In addition, one could hope to observe highly energetic quantized disclinations with strange multiplicities – in the absence of a manifest nematic order it is very hard to force them in by external means. It makes us wonder, could it be that states of this kind are around both in the classical- and the quantum context, being overlooked by experimentalists because of the lack of awareness?

The other novelty is the ordered nematic state which we found to behave more like a smectic (the 'glide smectic' of Section 8). This is a state having no analogy elsewhere. It rests on the unique dynamical principle of glide which on the quantum level drives a literal dynamical compactification, a reduction of dimensionality driven by spontaneous symmetry breaking. At the same time, since this state will be destroyed immediately by the presence of interstitials, it is not of relevance to condensed matter physics. As with more issues raised by this work, we wonder if this mechanism might be of use in the context of fundamental physics (cosmology and high energy physics). We will return to this issue later.

Let us now turn to the hard question is. Is our 'order superconductivity' of relevance to nature? Fact is that no experimental evidence is available supporting any of the hard predictions presented in this paper (the quantum nematic states, the spectrum of the superfluid, the electromagnet screening current oscillations). One statement can be made easily: if it exists, it is not particularly abundant and not easily accessible by condensed matter experimentation. As we emphasized repeatedly, all what is needed is a Bosonic quantum fluid characterized by a shear penetration depth which is large compared to the lattice constant. For detailed microscopic reasons such a condition is not easily satisfied starting out with electrons and/or atoms. There is no doubt that conventional (BCS) superconductors, as well as the Bose–Einstein atomic gases approach the ideal quantum gas limit very closely. ^{4}Helium is in principle a more fruitful territory. As we already discussed, the roton might be viewed as a first signal of the approach in the direction of the order superfluid, although it is still far removed from the massive shear modes we discussed in Section 6. An open issue is how matters look like in Helium layers. Reduction of dimensionality surely helps 'our' physics and after all there is an indication for a second phase transition suggesting the presence of hexatic order [17,18]. One would like to measure the excitation spectrum using inelastic neutron scattering to look for the massive shear modes, but this is obviously a very hard experiment.

Natural candidates for the order superconductivity are the under-doped high T_c superconductors. Evidences have been accumulating that the superconducting state is in a tight competition with a (anisotropic) 'Wigner crystal' (stripe phase) ([25] and references therein). A serious concern is that this stripe phase is itself formed because of strong lattice commensuration effects. It is a state which is uniquely associated with doping the Mott-insulator. The Mott-insulator itself is best understood as an electronic density wave which is commensurately pinned by the lattice. The stripe phase can be seen as a higher order commensurate state with the charges stripes to be interpreted as the discommensurations forming a lattice [43]. A case can be made that due to fluctuations the effects of the lattice potential can be much weakened so that at long wavelength the stripe phase might closely approach an effective Gallilean invariance [24,44]. If this is the case, dislocations might become cheap as compared to interstitials, and the conditions might be right for the emergence of quantum nematics. Although controversial, recent STM work by Kapitulnik and coworkers [45] is suggestive of a strongly dislocated character of a partially pinned stripe phase. There is also counter-evidence. The recent observation of striped 'halos' surrounding the vortex cores in the type II state of the cuprate superconductors has added credibility to the notion that the superconductor is in a tight competition with the stripe phase [46,47]. However, these data are quite successfully explained in terms of theories based on supersolids [31,32]. In fact, from the present work another counter argument follows. As we discussed in Section 4, in the quantum nematic state one would invariably expect the vortices to have strongly elongated shapes, and this is not observed.

Although the focus of this paper has been on condensed matter physics, a next open question is to what extent our findings can be of use in the context of fundamental theories, either in a metaphorical- or even in a literal sense. We already referred to the intriguing connections between generalized elasticity and gravity, especially promoted by Kleinert [2,39,40]. The theory can be reformulated in the language of differential geometry, starting with the identification of the metric $g_{ij} = \delta_{ij} + w_{ij}$. It can then be demonstrated that dislocations and disclinations take the roles of torsion- and curvature sources, respectively. A first requirement for the identification of space–time with an elasticity-type field theory is that the latter should be manifestly Lorentz-invariant, at least at long distances. This implies that one has to consider isotropic elasticity in $\mathrm{D}+1$ dimensions, such that the physics along the time axis is the same as that on the time slice. Amusingly, the relativistic generalization of the Coulomb nematic state as discussed in this paper turns out to be indistinguishable at large wavelength from the theory of general relativity [41]! A crucial ingredient is that in the relativistic medium the glide principle cannot exist. The reason is that, say, in three classical dimensions dislocation lines can have arbitrary orientations relative to their Burgers vectors (screw- versus edge dislocations). It is a fundamental requirement for glide that a time axis can be singled out. As we explained in Section 3, glide is fundamentally related to the neutrality of dislocations relative to compression. When glide is absent, dislocations will carry compression charge, and the result is that in the relativistic Coulomb nematic both shear and compression acquire a Higgs mass. The resulting medium is one which is only carrying curvature rigidity and such a 'liquid crystal' has the same long-wavelength properties as the space–time according to Einstein.

Our present work suggest some other alleys which might be worth investigating further with an eye on problems in fundamental physics. We already mentioned the intriguing feature that in the 'glide smectic' a dimension is compactified in the stress-photon universe due to the conspiracy of spontaneous breaking of the rotations in the embedding space, and the glide principle. One of the great unsolved problems in string theory is dynamical compactification. Our mechanism is stand alone in the sense that it is field-theoretical. It makes us wonder if some appropriate generalization of the basic mechanism in the much richer context of string theory can be formulated.

A final issue is the Higgs mechanism of high energy physics, believed to be responsible for the origin of non-radiative rest mass. It is incorporated in the Standard Model in the form of an effective field of the Ginzburg–Landau–Wilson type. A wide open question is, is this field fundamental or does it parameterize a collective property of, say, Planck scale matter? The only analogy available used to be the Bose-gas, and the notion that the Higgs field is coming from a Bose-gas of particles living at the Planck scale is for good reasons not very popular. Our 'order superconductor' offers an alternative perspective. It shows that the Higgs mass of the gauge fields is a secondary effect, driven by a primary condensation associated with symmetry restoration. Given that the latter has intriguing connections with the structure of space–time [41], it suggest an unexpected potential connection between the Higgs phenomenon and gravity.

Acknowledgments

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Appendix A. Abelian–Higgs duality in 2 + 1D

The duality discussed in this paper is closely related to the Abelian–Higgs duality. This is an important duality which has been studied in great detail and has many applications (see, e.g. [4–10]). Here we will just review the essence of this duality. For an even more minimal exposition, we refer to a paper by Zee [48].

Consider the Hamiltonian describing, e.g., a 2D array of Josephson junctions in the absence of dissipation and neglecting the coupling to electromagnetic fields (in fact, this describes a neutral superfluid),

$$H = Q\sum_i (n_i)^2 + J\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j), \tag{A.1}$$

where the second term describes the Josephson coupling (ϕ_i) is the superfluid phase on island i and the first term the 'charging' energy (n_i is number density). This is quantized because of number-phase conjugation $[n_i, \phi_j] = \mathrm{i}\delta_{ij}$. The Lagrangian is, in relativistic short hand, setting the phase velocity 1,

$$\mathcal{L} = \frac{1}{g}\left(\partial_\mu(\phi + \phi_V)\right)^2, \tag{A.2}$$

where the coupling constant $g = Q/J$ while ϕ is a (compact) O(2) field: $\phi \to \phi + 2\pi n$, $n \in Z$. This implies that in principle the phase field can have multivalued configurations, and these are lumped in ϕ_V. One can make this further explicit by the Villain construction $\phi_V = 2\pi n$, treating ϕ as non-compact and including the ns in the path integral measure. With some effort it can be shown that this regularization of the *XY* problem is qualitatively correct and quantitatively quite accurate [7].

It is obvious that a quantum-phase transition will occur. For $g \ll 1$ ϕ will order, describing the superfluid while for large g number fluctuations are suppressed and a state with a fixed particle number per site is a Mott-insulator, see Fig. 13.

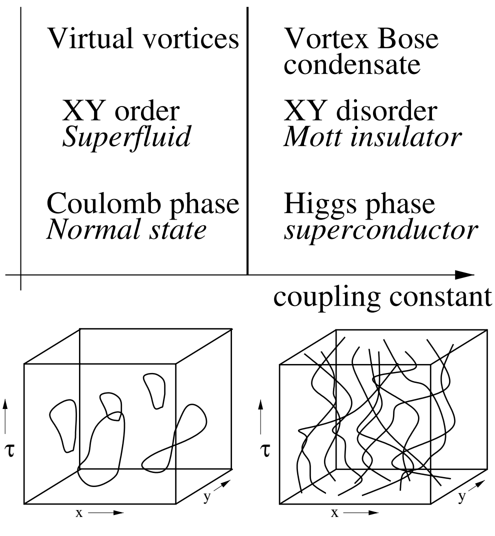

Fig. 13. Illustration of the physical meaning of the Abelian–Higgs duality and the various interpretations one can ascribe to the states on both sides of the duality. On the 'ordered' side vortices form closed loops in space–time, and these have blown out on the disordered side forming a tangle corresponding with a Bose condensate. One can either interpret the ordered side as a neutral superfluid (disordered side a Mott-insulator) or the disordered side as a superconductor (ordered side normal state with massless photons).

The topological excitation of superfluid order in 2 + 1 dimensions is a particle, the well known 'vortex pancake.' It carries a topological invariant on the time-slice defined by the circuit integral,

$$\oint \mathrm{d}\phi = 2\pi N, \tag{A.3}$$

where N is the winding number, or vorticity. Since the winding number is conserved, the vortex particle corresponds with a worldline in space–time. Using Stokes theorem, a conserved vortex current is derived which expresses the non-integrability of the field ϕ, collected in the piece ϕ_V,

$$J^V_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\phi_V, \tag{A.4}$$

$$\partial_\mu J^V_\mu = 0, \tag{A.5}$$

where the time component (J_0) corresponds with the vortex density. Introducing an auxiliary field ξ_μ, Eq. (A.2) becomes after a Hubbard–Stratanowich transformation,

$$\mathcal{L} = g\xi_\mu\xi_\mu + \mathrm{i}\xi_\mu\partial_\mu(\phi + \phi_V) \tag{A.6}$$

and it is observed that in this stage the coupling constant g has already been inverted, meaning that weak coupling in ϕ is strong coupling in ξ_μ and vice versa. Since ϕ is by definition integrable the derivative can be shifted,

$$L = g\xi_\mu\xi_\mu + \mathrm{i}\xi_\mu\partial_\mu\phi_V + \mathrm{i}\xi_\mu\partial_\mu\phi, \tag{A.7}$$

$$= g\xi_\mu\xi_\mu + \mathrm{i}\xi_\mu\partial_\mu\phi_V - \mathrm{i}\phi\partial_\mu\xi_\mu \tag{A.8}$$

ϕ appears as a Lagrange multiplier and upon integration it follows that ξ_μ is conserved and it can therefore be written as the curl of an 'emerging' $U(1)$ gauge field $\vec{A}$,

$$\partial_\mu\xi_\mu = 0, \qquad \xi_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \tag{A.9}$$

this describes the conservation of the superfluid current. Inserting Eq. (A.9) in the remaining part of Eq. (A.8),

$$L = gF_{\mu\nu}F^{\mu\nu} + \mathrm{i}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda\partial_\mu\phi_V = gF_{\mu\nu}F^{\mu\nu} + \mathrm{i}A_\mu J^V_\mu, \tag{A.10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the field strength of the $\vec{A}$ field, while the second line follows from the first modulo a boundary term.

Hence, the vortex particle as described by the current Eq. (A.4) acts like a source for the superfluid gauge-field and the latter behaves like a electromagnetic field on the vortex particle! Reversely, one might also want to claim that quantum-electromagnetism is nothing else than an ordered XY magnet in disguise. The only reason for this to be special for 2 + 1D is that in this dimension vortices are like particles. In 3 + 1D vortices become strings and this is somewhat hard to reconcile with the point-like character of the particles.

The hard work has been done and the remainder rests on standard notions of field-theory. Vortices are sources of kinetic energy (charging energy in the phase-dynamics interpretation). Upon increasing the coupling constant, closed loops of vortex lines will appear (virtual fluctuations) and these loops will steadily increase their characteristic size. When the coupling constant exceeds a critical value, a 'loop blow out' transition will occur. The vortices turn into real particles while their worldlines will entangle forming a Bose condensate. Eq. (A.10) describes a single vortex-worldline and in the disordered phase one is dealing with an interacting system of bosonic vortex particles. This system is just a system of bosons which is charged relative to an electromagnetic field, and in the neighborhood of the quantum phase transition this is described in terms of a Ginzburg–Landau–Wilson action,

$$\mathcal{L}_{\text{dual}} = |(\partial_\mu - \mathrm{i}A_\mu)\Phi^V|^2 + m^2|\Phi^V|^2 + w|\Phi^V|^4 + gF_{\mu\nu}F^{\mu\nu}. \tag{A.11}$$

This disorder field theory can be derived explicitly (see Appendix B). The mass term ($m^2|\phi^V|^2$) corresponds with the core-energy/meandering kinetic energy balance of the vortex loops. The proliferation of vortices is encoded by m^2 changing sign. The Φ^4 term represents the short range (hard-core) repulsions while the coupling to the auxiliary gauge field describes the long-range Coulombic interactions between the vortices as mediated by the superfluid fields. Eq. (A.11) describes a charged superfluid and for $m^2 < 0$ (the phase disordered state) this enters the Meissner phase where the superfluid 'photon' is expelled. Hence, there are no low lying excitations and this superconducting condensate of vortex-bosons describes precisely the Mott-insulator!

Appendix B. Glide constraints, loop gases, and GLW field theory

It is a central result in quantum-field theory and statistical physics that the Ginzburg–Landau–Wilson 'ϕ^4' theory describes an ensemble of fluctuating, interacting lines. In the present context we are dealing with the special circumstance that the dislocation worldlines can only meander in planes spanned by the Burgers vectors of the dislocations and the time direction, due to the glide constraint. In Section 3.3 we use Landau's original argument to deduce the structure of the field theory describing the dislocation condensate, postulating that the gradient terms follow from a conjugate momentum acting on a 'wavefunction.' From this argument it follows that the condensate field is just a conventional one, while the effects of glide are entirely absorbed in the couplings to the stress gauge fields. Starting from a loop gas perspective, this is less obvious. Why should the condensate be described by a 2 + 1D GLW field, while the dislocations might even decide to move in just one direction? Shouldn't this be an effective 1 + 1D problem? Such an expectation is based on a flawed understanding of the loop gas—GLW mapping: the order parameter field Ψ just keeps track of the fact that dislocation loops can appear everywhere in space–time, and only the couplings to the gauge fields keep track of the direction of the motions. To make this explicit, we will first shortly review the standard loop gas-GLW mapping, to subsequently analyze the (exaggurated) case that Burgers vectors can only point along x and y directions.

The standard mapping is discussed in great detail in the first part of the Kleinert volumes [7], and here we review just the essence of the mapping using a statistical physics language. Defect–anti-defect worldline loops behave like random walkers and it is convenient to define these on a lattice. The partition function for one closed loop is given by

$$Z_1 = \sum_{\mathbf{x},N} \frac{(2D)^N}{N} P(\mathbf{x},\mathbf{x},N) \mathrm{e}^{-\beta\epsilon N} \tag{B.1}$$

ϵ is the core energy (energy/unit length) of the defect, β inverse temperature (coupling constant in the quantum case). The factor $(2D)^N$ (D is space–time dimensionality) accounts for the number of spatial configurations of a chain of length N and the factor N^{-1} ensures that one particular loop is only counted once. $P(\mathbf{x},\mathbf{x},N)$ is the *return probability*, i.e., the probability that the random walker returns to its point of departure so that it forms a closed loop. This probability is governed by a discrete diffusion equation (a is the lattice constant, $\bar{\Delta}_\mu$, Δ_μ backward- and forward finite difference operators, respectively),

$$\bar{\Delta}_N P(\mathbf{x},\mathbf{y},N) = \frac{a}{2D} \sum_\mu \bar{\Delta}_\mu \Delta_\mu P(\mathbf{x},\mathbf{y},N-1) \tag{B.2}$$

with the boundary condition $P(\mathbf{x},\mathbf{y},0) = \delta_{\mathbf{x},\mathbf{y}}$. This is solved by the Fourier ansatz $P(\mathbf{K},N) = P(\mathbf{K})^N$,

$$P(\mathbf{k}) = 1 - \frac{1}{2D} \sum_\mu \left(\mathrm{e}^{\mathrm{i}k_\mu a} - 1\right)\left(\mathrm{e}^{-\mathrm{i}k_\mu a} - 1\right) \to 1 - \frac{a^2}{2D} \sum_\mu k_\mu^2. \tag{B.3}$$

For future use, notice that the gradient terms appearing in the field theory can be traced back to this return probability; the gradients appear because the worldlines have to form loops.

Using Eq. (B.3) and the identity $\sum_{N=1}^\infty x^N/N = -\ln(1-x)$, the one loop partition function Eq. (B.1) becomes,

$$Z_1 = -\sum_{\mathbf{k}} \ln\left(1 - 2DP(\mathbf{k})\mathrm{e}^{-\beta\epsilon}\right). \tag{B.4}$$

The grand canonical partition function for a gas of loops is obtained by exponentiation of the one loop partition sum,

$$\Omega = \mathrm{e}^{Z_1} = \Pi_{\mathbf{k}} G_0(\mathbf{k}), \tag{B.5}$$

$$G_0(\mathbf{k}) = \frac{1}{1 - \mathrm{e}^{-\beta\epsilon} 2DP(\mathbf{k})}. \tag{B.6}$$

We now observe that the product of propagators can be reproduced by Gaussian integration over complex scalar fields $\phi(\mathbf{x})$,

$$\Omega = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left\{ -\frac{1}{2} \int \mathrm{d}^D x \phi^*(\mathbf{x}) G_0(\mathbf{x})^{-1} \phi(\mathbf{x}) \right\} \tag{B.7}$$

while the inverse propagator becomes in the continuum limit,

$$G_0(\mathbf{x})^{-1} = \partial_\mu^2 + m^2 \tag{B.8}$$

with a mass

$$m^2 = a^{-2}\left(\mathrm{e}^{\beta\epsilon} - 2D\right) \tag{B.9}$$

which is changing sign when the meandering entropy is overcoming the core energy. We have recovered the famous result that Gaussian field theory is just equivalent to a gas of random walking closed loops.

In the context of this paper, the special circumstance is that due to the glide constraint the defect-worldlines cannot freely meander through the embedding space–time. Instead, the loops are oriented in the 2D planes spanned by the Burgers vectors of the (anti)dislocations and the imaginary time axis. In the main text (Section 3.3) we follow the strategy of Landau to find that this 'reduced dimensionality' effect is not reflected in the dimensionality of the matter fields but only in the couplings to the gauge fields. Can this be understood explicitly in terms of the mapping loop gas-field theory?

Let us consider the following loop gas problem which is clearly representative for the problem. Consider a cubic lattice $\mathbf{x} = (x_i, y_i, \tau_i)$ (dimension $D = 3$) and assume that loops of random walkers occur which are exclusively meandering in either the (x, τ) or (y, τ) planes (dimension $D' = 2$) with the same a priori probability. The generalization of the one loop partition function Eq. (B.1) is,

$$Z_1 = \sum_{\mathbf{x},N} \frac{(2D')^N}{N} \left(P_x(\mathbf{x},\mathbf{x},N) + P_y(\mathbf{x},\mathbf{x},N)\right) \mathrm{e}^{-\beta\epsilon N} \tag{B.10}$$

where $\mathbf{x}$ refers to all points on the space–time lattice. The reduced dimensionality is reflected only in the factor $(2D')^N$ and the return probabilities $P_\alpha(\mathbf{x},\mathbf{x},N)$ referring now to probability of return by meandering exclusively in the (α, τ) planes, with $\alpha = x, y$. These are given by Eq. (B.3) but now restricted to the meandering planes. Explicitly (ω is Matsubara frequency),

$$\begin{aligned} P_\alpha(\mathbf{k}) &= 1 - \frac{1}{2D'} \left[\left(\mathrm{e}^{\mathrm{i}k_\alpha a} - 1\right)\left(\mathrm{e}^{-\mathrm{i}k_\alpha a} - 1\right) + \left(\mathrm{e}^{\mathrm{i}\omega a} - 1\right)\left(\mathrm{e}^{-\mathrm{i}\omega a} - 1\right) \right] \\ &\to 1 - \frac{a^2}{2D'} \left[k_\alpha^2 + \omega^2 \right] \end{aligned} \tag{B.11}$$

and the one loop partition function becomes (compare Eq. B.4),

$$Z_1 = -\sum_{\mathbf{k}} \sum_{\alpha=x,y} \ln\left(1 - 2D' P_\alpha(\mathbf{k}) \mathrm{e}^{-\beta\epsilon}\right). \tag{B.12}$$

The grand partition function of the loop gas is as before obtained by exponentiation,

$$\Omega = e^{Z_1} = \Pi_{\mathbf{k}} G_0(\mathbf{k}), \tag{B.13}$$

$$G_0(\mathbf{k}) = \Pi_{\alpha=x,y} \frac{1}{1 - \mathrm{e}^{-\beta\epsilon} 2D' P_\alpha(\mathbf{k})}. \tag{B.14}$$

We now observe that in the continuum limit,

$$\begin{aligned} G_0(\mathbf{k}) &\sim \frac{1}{(m')^2 + k_x^2 + \omega^2} \frac{1}{(m')^2 + k_y^2 + \omega^2} \\ &\sim \frac{1}{(m')^2} \frac{1}{(m')^2 + k_x^2 + k_y^2 + 2\omega^2 + \mathrm{O}(k^4)}. \end{aligned} \tag{B.15}$$

In the continuum limit factors $O(k^4)$ can be neglected and we observe that the partition function is given in terms of propagators living in the full embedding space–time. The gradient terms find their origin in the return probabilities and given that the loops have the same a priori probability to occur both in x and y directions, both add gradients to the propagator. The effect of the loops meandering in a reduced dimensionality just ends up in a parametric change in the mass reflecting the reduction of the meandering entropy,

$$(m')^2 = \frac{e^{\beta\epsilon} - 2D'}{a^2} \tag{B.16}$$

and a renormalization of the velocity by a factor $\sqrt{2}$ due to the fact that the time axis is shared. Since the propagators are $2+1$D, the fields needed to reproduce the propagators are also $2+1$D,

$$\Omega \sim \int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left\{-\frac{1}{2}\int d^d r\, d\tau\left(|\partial_x\phi|^2 + |\partial_y\phi|^2 + \frac{1}{2}|\partial_\tau\phi|^2 + (m')^2|\phi|^2\right)\right\}. \tag{B.17}$$

This demonstrates that the assertion in the main text regarding the nature of the effective field theory describing the dislocation condensate is correct. The field ϕ describes the statistical distributions of dislocation–anti-dislocation loops in space–time and this is a $2+1$D problem. The consequence of the glide principle, that the kinetic energy is oriented along the Burgers vectors, does not change the dimensionality of the field but it does have consequences for the screening of the stress photons. The oriented nature of the kinetic energy becomes manifest when screening currents are required for the shear-Meissner effect.

Appendix C. Stress gauge fields in the classical 2D limit

Here we rederive the stress gauge field action for the classical 2D limit, corresponding with the high temperature limit of the $2+1$D theory (see also [2, vol. II, Section 4.6]).

In two space dimensions the time axis is absent and one is forced to pick a Coulomb gauge, setting the space components of the stress gauge fields to zero, keeping only the time components. The stress gauge fields are defined through $\sigma^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda$. Keeping only the B^a_τ fields and calling these A^a to get contact with Kleinert's notation, and ignoring time derivatives: $\sigma^x_x = \partial_y A^x, \sigma^y_y = -\partial_x A^y, \sigma^y_x = \partial_y A^y, \sigma^x_y = -\partial_x A^x$. We recognize directly Kleinert's fields and the free energy is,

$$S_{2D} = \frac{1}{4\mu}\int d^2x\left[(\sigma^x_x)^2 + 2(\sigma^x_y)^2 + (\sigma^y_y)^2 - \frac{\nu}{1+\nu}(\sigma^x_x + \sigma^y_y)^2\right] \tag{C.1}$$

after inserting the A^a fields, Fourier transforming and using the standard space longitudinal/transversal composition,

$$A^x = \hat{q}_x A^L + \hat{q}_y A^T, \qquad A^y = \hat{q}_y A^L - \hat{q}_x A^T. \tag{C.2}$$

The symmetry condition $\sigma^x_y = \sigma^y_x$ has to be imposed which becomes $\partial_a A^a = 0$, implying that the A^L fields are unphysical. A simple calculation yields for the 2D action,

$$S_{2D} = \frac{1}{4\mu(1+\nu)} \int d^2q\, q^2 |A^T|^2 \tag{C.3}$$

showing that 2D classical matter is governed by a single stress photon mediating in fact exclusively shear stresses. Photons carrying compression require the existence of a time axis, see the derivation of the compression action Eq. (47) in Section 3.

Appendix D. Relating the Goldstone propagators to the dual photons

In this paper we use repeatedly the propagators of the stress photons to compute the propagators and spectral functions in the Goldstone-sector. These are related by simple general expressions, which we will derive in this appendix. For convenience, we specialize to quantum elasticity but the reader will recognize that the derivation is quite general.

Consider the action of quantum-elasticity S_0. Define a generating functional by adding a source $\mathcal{J}^a_\mu$ coupling to the strain fields,

$$G(\mathcal{J}) = \frac{1}{Z} \int \mathcal{D}u^a \exp\left(- S_0 - \int d\Omega \left[\mathcal{J}^a_\mu \partial_\mu u^a \right] \right). \tag{D.1}$$

The elastic (or strain) propagator is defined through the functional derivatives of the generating functional by the sources ($\vec{r}$ is space–time coordinate),

$$\begin{aligned} \langle\langle \partial_\mu u_a(\vec{r}_1) | \partial_\nu u_b(\vec{r}_1) \rangle\rangle &\equiv \left[\frac{\delta^2 G(\mathcal{J})}{\delta \mathcal{J}^a_\mu(\vec{r}_1) \delta \mathcal{J}^b_\nu(\vec{r}_2)} \right]_{J=0} \\ &= \frac{1}{Z} \int \mathcal{D}u^a (\partial_\mu u_a(\vec{r}_1))(\partial_\nu u_b(\vec{r}_2)) \exp(-S_0). \end{aligned} \tag{D.2}$$

The strategy is to dualize the generating functional, to subsequently take the functional derivatives to the sources $\mathcal{J}$ using the action expressed in the dual stress Gauge fields. Introduce auxiliary stress-fields, so that the strain action S_0 turns into the stress action $S_{\text{dual}}(\sigma^a_\mu)$. Keeping track of the source term,

$$\begin{aligned} G(\mathcal{J}) &= \frac{1}{Z} \int \mathcal{D}\sigma^a_\mu \mathcal{D}u^a \exp\left(- S_{\text{dual}}(\sigma^a_\mu) - \int d\Omega \left(2i\sigma^a_\mu + \mathcal{J}^a_\mu \right) \partial_\mu u^a \right) \\ &= \frac{1}{Z} \int \mathcal{D}\sigma^a_\mu \mathcal{D}u^a \exp\left(- S_{\text{dual}}(\sigma^a_\mu) + \int d\Omega\, i u^a \partial_\mu \left(2\sigma^a_\mu - \mathcal{J}^a_\mu \right) \right) \\ &= \frac{1}{Z} \int \mathcal{D}\sigma^a_\mu \mathcal{D}u^a \delta\left(2\sigma^a_\mu - i\mathcal{J}^a_\mu \right) \exp(-S_{\text{dual}}). \end{aligned} \tag{D.3}$$

The constraint can be resolved by introducing the stress gauge fields,

$$\sigma^a_\mu - \frac{i}{2}\mathcal{J}^a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda, \qquad \sigma^a_\mu = \left(\epsilon_{\mu\nu\lambda}\partial_\nu B^a_\lambda + \frac{i}{2}\mathcal{J}^a_\mu \right). \tag{D.4}$$

The dual action has the form $S_{\mathrm{dual}} = \int \mathrm{d}\Omega\, \sigma^a_\mu C^{-1}_{\mu\mu' ab} \sigma^b_{\mu'}$ and inserting Eq. (D.4) we obtain for the generating functional,

$$G(\mathcal{J}) = \frac{1}{Z} \int \mathcal{D}B^a_\mu \delta(\partial_\mu B^a_\mu) \exp\left(- S_{\mathrm{dual}}(B^a_\mu) - \int \mathrm{d}\Omega \left[\mathrm{i}\mathcal{J}^a_\mu C^{-1}_{\mu\mu' ab} \epsilon_{\mu'\nu\lambda} \partial_\nu B^b_\lambda - \frac{1}{4} \mathcal{J}^a_\mu C^{-1}_{\mu\mu' ab} \mathcal{J}^b_{\mu'} \right] \right), \tag{D.5}$$

where $S_{\mathrm{dual}}(B^a_\mu)$ is just the 'stress Maxwell' action. Eq. (D.5) is of the required form to deduce the relation between strain- and stress photon propagators. Use Eq. (D.2) but now calculate the functional derivatives using Eq. (D.5). Reinsert the σs for notational convenience,

$$\langle\langle \partial_\mu u_a(\vec{r}_2) | \partial_\nu u_b(\vec{r}_1) \rangle\rangle = \delta(\vec{r}_1 - \vec{r}_2) \delta_{\mu,\nu} \delta_{a,b} C^{-1}_{\mu\mu aa} - C^{-1}_{\mu\lambda ac} C^{-1}_{\nu\kappa bd} \langle \sigma^c_\lambda(\vec{r}_1) | \sigma^d_\kappa(\vec{r}_2) \rangle, \tag{D.6}$$

where by definition

$$\langle\langle \sigma^c_\lambda(\vec{r}_1) | \sigma^c_\lambda(\vec{r}_2) \rangle\rangle \equiv \frac{1}{Z} \int \mathcal{D}\sigma^a_\mu \delta(\partial_\mu \sigma^a_\mu) \sigma^c_\lambda(\vec{r}_1) \sigma^c_\kappa(\vec{r}_2) \exp\left(- S_{\mathrm{dual}}(\sigma^a_\mu) \right), \tag{D.7}$$

i.e., the stress propagator which has to be averaged using the dual action. In frequency–momentum space this becomes,

$$\langle\langle p_\mu u_a(-\vec{p}) | p_\nu u_b(\vec{p}) \rangle\rangle = C^{-1}_{\mu\mu aa} - C^{-1}_{\mu\lambda ac} C^{-1}_{\nu\kappa bd} \langle\langle \sigma^c_\lambda(-\vec{p}) | \sigma^d_\kappa(\vec{p}) \rangle\rangle \tag{D.8}$$

which is the starting point for the computations in the main text. The algorithm is simple: associate the strains with the stresses via the stress-strain relations $p_\mu u_a(\vec{p}) = C^{-1}_{\mu\lambda ac} \sigma^c_\lambda(-\vec{p})$ and the stress–stress propagators $\langle\langle \sigma | \sigma \rangle\rangle$ can be computed using the stress gauge fields. The above derivation shows that the Goldstone propagator $\langle\langle \partial_\mu u^a | \partial_\nu u^b \rangle\rangle$ is proportional to a constant minus the dual photon propagators. This is a general result, since one relates the propagator of a scalar Goldstone boson with the field-strength propagators $\sim \epsilon_{\mu\nu\lambda} A_\lambda$ of dual gauge fields which are transversal. We leave it as an exercise for the reader to demonstrate that in the case of the simple XY-electromagnetism duality discussed in Appendix A the relations are as follows: the spin wave propagator is $\langle\langle p_\mu \phi | p_\nu \phi \rangle\rangle \sim p_\mu p_\nu / |p|^2$. The field strength propagators $\langle\langle \xi_\mu | \xi_\nu \rangle\rangle \sim \delta_{\mu\nu} - p_\mu p_\nu / |p|^2$, obeying the transversality condition. Hence, in agreement with the above, $\langle\langle p_\mu \phi | p_\nu \phi \rangle\rangle = \delta_{\mu\nu} - \langle\langle \xi_\mu | \xi_\nu \rangle\rangle$.

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