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Note on a theorem of Dr. Cherry about uniform integrals of the dynamical equations,

by *J. Woltjer Jr.*

In two papers recently published *) Dr. CHERRY has demonstrated the existence of uniform integrals for differential equations of certain standard type, and made an application of his results to POINCARÉ's theorem of the non-existence of uniform integrals of the differential equations of astronomical mechanics. The purpose of this note is to offer some remarks on Dr. CHERRY's theorems.

I. The fundamental theorem of Dr. CHERRY may be stated as follows:

The system of differential equations:

$$(1) \qquad \frac{dx_i}{dt} = X_i, \qquad i = 1, \ldots n$$

admits $n-1$ integrals, that can be developed as power-series in $x_1, \ldots x_n$, if the X_i can be developed in the same way and *are not all reduced to zero* for $x_1 = \ldots x_n = 0$.

This theorem may easily be reduced to the ordinary existence-theorems of differential equations **) in the following way.

Suppose e.g. that X_1 is not reduced to zero if $x_1 = \ldots x_n = 0$; then x_1 may be used as independent variable; the equations (1) are reduced to:

$$(2) \qquad \frac{dx_i}{dx_1} = P_i, \qquad i = 2, \ldots n, \qquad \frac{dt}{dx_1} = P_1,$$

the P_i being power-series in $x_1, \ldots x_n$. That solution of the first $n-1$ equations (2), which is reduced to $x_i = c_i$ if $x_1 = 0$, can be developed as a power-series in x_1 and $c_2, \ldots c_n$. Consequently we have:

$$(3) \qquad x_i = c_i + \Pi_i, \qquad i = 2, \ldots n,$$

the Π_i being power-series in x_1 and $c_2, \ldots c_n$ without constant terms, reducing to zero for $x_1 = 0$. The solution of (3) for c_i furnishes the desired integrals:

$$(4) \qquad c_i = F_i(x_1, \ldots x_n),$$

where F_i are power series in $x_1, \ldots x_n$.

II. The application of this theorem to the equations of Hamilton:

$$(5) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i},$$

where F is a power-series in $x_1, \ldots x_n, y_1, \ldots y_n$, lead Dr. CHERRY, through the canonical transformation:

$$(6) \qquad x_i = \sqrt{2\xi_i}\cos\eta_i, \qquad y_i = \sqrt{2\xi_i}\sin\eta_i,$$

to the conditions of POINCARÉ's theorem.

However the essential condition for the application of the theorem I is that F contains terms that are linear in x_i and y_i.

Now I am not aware of one problem of astronomical mechanics, that is important for the question under discussion, in which this condition is satisfied. I refer to two problems:

a. the secular variations of the elliptical elements may be determined from equations of the form (5), but F then is at least of the second degree in x_i and y_i;

b. the oscillations about an equilibrium solution (the Trojans) may be determined from equations of the form (5), but F then is at least of the second degree in x_i and y_i.

So I cannot see how the application of Dr. CHERRY's theorem may furnish uniform integrals of astronomical problems.

*) *Proceedings of the Cambridge Philosophical Society*, 22, 3.
**) H. POINCARÉ, *Méthodes Nouvelles* I, Chap. II.