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On excitation and maintenance of secondary oscillations in pulsating stars,

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Introduction. The differential equations for the components of the radial pulsation in non-adiabatic motion have been developed in a paper published several years ago [2]). Generally, a simple-periodic oscillation and its maintenance appeared possible. If special circumstances prevailed, a permanent secondary oscillation might be excited: however, the restrictions imposed on the analytical treatment (the limited interaction of only two degrees of freedom) allowed no quantitatively-satisfactory solution. An analysis, in certain respects more general, of the anomalous phase-relation between first and second harmonic in the radial velocity of ζ Geminorum [3]) removed these restrictions and showed the possibility of a secondary permanent oscillation of very long period in the pulsation of this variable.

The theoretical developments contained in the present paper are general: the limitation of the energy-function to some terms supposed to be of principal importance, though still illuminating many aspects of the problem, is not necessary; hence, the structure of the simple-periodic solution is more clearly apparent. The analysis of the secondary oscillations shows many possibilities to exist: a pulsation of very long period comparable with that considered in the case of ζ Geminorum; an oscillation of arbitrary period comparable with those observed in cluster-type variables and long-period variables; and also the coexistence of both.

1. *The equations of motion and the conditions required for a permanent multiple-periodic solution.*

The partial differential equation of the non-adiabatic radial motion of a star may be broken up into the system of simultaneous ordinary differential equations, infinite in number:

$$\frac{dJ_i}{dt} = -2\alpha_i J_i - \frac{\partial H}{\partial w_i}, \quad \frac{dw_i}{dt} = \frac{\partial H}{\partial J_i} \qquad i = 1, 2, \ldots;$$

$4\pi H$, the sum of kinetic, gravitational and internal energy, is a function of the variables J_i, w_i only, independent of the time t; these variables are related to the quantities C_i and $\frac{dC_i}{dt}$ by the equations

$$C_i = \sqrt{\frac{2J_i}{n_i}} \cos w_i, \quad \frac{dC_i}{dt} = -\sqrt{2 n_i J_i} \sin w_i;$$

these quantities determine the radius-vector r, related to the value r_n in a "normal" static stellar structure by the series:

$$\frac{r - r_n}{r_n} = \sum_1^\infty C_i s_i(r_n);$$

the functions $s_i(r_n)$ are normalised solutions of the ordinary linear differential equation that determines the infinitesimal oscillations and their periods $\frac{2\pi}{n_i}$; the coefficients α_i are the damping-constants; their introduction in this way into the differential equations is a simplification, which does away with the implicit-time-dependence of the function H in a still more general treatment.

Suppose a multiple-periodic solution of these equations to exist, consisting of periodic functions with period 2π of phase-arguments ω_k that are independent linear functions of the time t. Then, multiplication of the equations respectively with $\frac{\partial w_i}{\partial \omega_k}$, $-\frac{\partial J_i}{\partial \omega_k}$, addition of the results and reduction of the resulting equations to the constant term in their goniometric developments, leads to the system of relations:

$$\sum_{i=1}^\infty \alpha_i \overline{J_i \frac{\partial w_i}{\partial \omega_k}} = 0 \qquad k = 1, 2, \ldots,$$

the bar denoting the mean value of the quantity concerned with regard to the variables ω_1, ω_2, ... The fact that this mean value is zero in the quantity

$$\sum_{i=1}^\infty \left(\frac{dJ_i}{dt}\frac{\partial w_i}{\partial \omega_k} - \frac{dw_i}{dt}\frac{\partial J_i}{\partial \omega_k}\right)$$

is evident, if J_i and w_i are replaced by their goniometric developments, supposed to exist, and account

[1]) See Note by the Editor on p. 136.
[2]) *B.A.N.* No. 303.
[3]) *B.A.N.* No. 359.

is taken of the fact that to this mean value only contribute terms of the same argument in J_i and w_i, which cancel out in the difference

$$\frac{\partial J_i}{\partial \omega_s}\frac{\partial w_i}{\partial \omega_k} - \frac{\partial w_i}{\partial \omega_s}\frac{\partial J_i}{\partial \omega_k} .$$

If the solution is simple-periodic these equations may be reduced to the one relation also necessary in the general case:

$$\sum_1^\infty \alpha_i \, \overline{J_i \frac{d w_i}{dt}} = 0 .$$

The quantities involved then are functions of one constant of integration corresponding to the amplitude of the oscillation; hence, this amplitude must have a definite value.

Generally the mean values of $J_i \frac{d w_i}{dt}$ $(i = 1, 2, \ldots)$ have all the same sign; hence at least one of the damping-constants must be negative; physical considerations allow the possibility $\alpha_1 < 0$.

2. *Transformation of the equations.*

The differential equations may be transformed by introduction of the variable w_1 as independent variable, a procedure well known in celestial mechanics; in this way, if the damping-constants are neglected, the then existing integral $H =$ constant reduces the order of the system.

Consider the relation $H = n_1 z$ and determine by this equation the variable J_1 as a function Q of $J_2, \ldots, w_1, w_2, \ldots, z$. Then the ratios

$$\frac{\partial H}{\partial w} : \frac{\partial H}{\partial J_1}, \; \frac{\partial H}{\partial J_s} : \frac{\partial H}{\partial J_1}, \; (s > 1) ,$$

introduced by division of the equations by $\frac{d w_1}{dt}$, are equal to $-\frac{\partial Q}{\partial w_s}, -\frac{\partial Q}{\partial J_s}$; the quantity $1 : \frac{\partial H}{\partial J_1}$ is equal to $\frac{1}{n_1}\frac{\partial Q}{\partial z}$. Hence, if the equations are restricted to the presence in the right-hand members of the derivatives of H only, the transformed equations are:

$$\frac{d J_i}{d w_1} = \frac{\partial Q}{\partial w_i}, \quad \frac{d w_i}{d w_1} = -\frac{\partial Q}{\partial J_i}, \; i > 1$$

$$\frac{dz}{d w_1} = 0 \quad , \quad \frac{d(-n_1 t)}{d w_1} = -\frac{\partial Q}{\partial z} .$$

The transformed damping-terms must still be added; these are the quantities to be added to the J_i equations

$$- 2 \alpha_i J_i \frac{dt}{d w_1};$$

the z-equation receives the addition

$$-\frac{2}{n_1} \sum_{i=1}^\infty \alpha_i J_i \frac{d w_i}{d w_1},$$

corresponding to the equation

$$\frac{dH}{dt} + 2 \sum_{i=1}^\infty \alpha_i J_i \frac{d w_i}{dt} = 0 .$$

Then, if a multiple-periodic solution exists, periodic with periods 2π in w_1 and the phase-arguments χ_i $(i = 1, 2, 3, \ldots)$ that are independent linear functions of w_1 and correspond in some way to the variables $n_1 t, w_2, \ldots$, the reduction applied in the preceding section leads to the system of necessary relations:

$$\sum_{i=2}^\infty \alpha_i \, \overline{J_i \left\{\frac{\partial w_i}{\partial \chi_k}\frac{d(n_1 t)}{d w_1} - \frac{\partial (n_1 t)}{\partial \chi_k}\frac{d w_i}{d w_1}\right\}} = \alpha_1 \overline{Q \frac{\partial (n_1 t)}{\partial \chi_k}}, \quad k = 1, 2, \ldots$$

3. *The solution of the equations of motion.*

Contrary to the procedure followed in the paper on ζ Geminorum[1]), first the damping-constants will be omitted in the equations of motion; then the corresponding solution is to be substituted in the relations necessary to maintain the oscillations and the values of the independent amplitudes may be determined. Then the reaction of the damping-constants in the solution should be taken into account and the approximation be continued.

In this restricted form the equations of motion are closely related to the differential equations of celestial mechanics.

The principal part of the analysis consists in the determination of a simple-periodic solution.

The function Q is periodic with periods 2π in the dependent variables $w_2, \ldots$ and the independent variable w_1; the coefficients of the corresponding goniometric development are functions of $z, J_2, \ldots$; the value of z is constant.

If only some critical periodic terms are considered, the construction of the simple-periodic solution is easily performed: it has been treated of in previous papers on the subject[1]). This restriction is not at all necessary; it only clearly shows the principal features. If the "normal" static stellar constitution is sufficiently known, the determination of the solution may be carried out to a higher degree of approximation without difficulty.

If the simple-periodic solution has been constructed it may be used for the determination of the secondary oscillations by a transformation to new dependent

[1]) *B.A.N.* No. 359.

[1]) *B.A.N.* No. 303 and 359.

variables. These variables X_i and x_i ($i > 1$) are related to the variables $J_2, \ldots, w_2, \ldots$ by the relations that equate the differences

$$\sqrt{2X_i}\cos x_i - \sqrt{2J_i}\cos w_i, \sqrt{2X_i}\sin x_i - \sqrt{2J_i}\sin w_i$$

to the values of $\sqrt{2J_i}\cos w_i$, $\sqrt{2J_i}\sin w_i$ in the simple-periodic solution, hence, to known functions of w_1, periodic in w_1 with period 2π. The resulting differential equations are:

$$\frac{dX_i}{dw_1} = \frac{\partial R}{\partial x_i}, \quad \frac{dx_i}{dw_1} = -\frac{\partial R}{\partial X_i}, \quad i > 1;$$

the function R is derived from Q by subtraction of the terms of order zero and one in the variables $\sqrt{2X_i}\cos x_i, \sqrt{2X_i}\sin x_i$; it is a multiple-periodic function of $w_1, x_2, x_3, \ldots$ with periods 2π, the coefficients depending on the X_i variables. If the solution has been constructed the value of $n_1 t$ is derived by a simple quadrature from the differential equation

$$\frac{d(n_1 t)}{dw_1} = \frac{\partial Q}{\partial z}.$$

The solution of the new equations is more complicated, as a simple-periodic solution is irrelevant; hence, a qualitative treatment is appropriate, reducing the function R to a single periodic term.

Suppose

$$R = R_\circ + R_1 \cos\psi,$$

ψ being equal to some selected linear combination with integral coefficients of $w_1, x_2, x_3, \ldots$, say $\psi = k_1 w_1 + k_2 x_2 + k_3 x_3$; k_2 and k_3 are supposed to be non-zero integers. Generally these restricted equations allow a solution $\psi = 0$ or π, suppose 0, the values of the variables $X_2, X_3, \ldots$ being constant and determined by the equation:

$$k_1 = k_2 \frac{\partial(R_\circ + R_1)}{\partial X_2} + k_3 \frac{\partial(R_\circ + R_1)}{\partial X_3}.$$

In this solution the variables $x_2, x_3, \ldots$ are linear functions of w_1, the mean motions being connected by the relation

$$k_1 + k_2 \frac{dx_2}{dw_1} + k_3 \frac{dx_3}{dw_1} = 0.$$

If $|k_2|$ or $|k_3|$ is equal to unity, then, as this generally makes R_1 proportional to $\sqrt{X_2}$ or $\sqrt{X_3}$, the equation of condition between the constant values of X_2 and X_3 readily may be solved; if not, special values, possibly very large, are involved.

The variations with regard to this solution are determined by the differential equations:

$$\frac{d\delta X_2}{dw_1} = -k_2 R_1 \delta\psi, \quad \frac{d\delta x_2}{dw_1} = -\frac{\partial^2(R_\circ + R_1)}{\partial X_2^2}\delta X_2 - \frac{\partial^2(R_\circ + R_1)}{\partial X_2 \partial X_3}\delta X_3,$$

$$\frac{d\delta X_3}{dw_1} = -k_3 R_1 \delta\psi, \quad \frac{d\delta x_3}{dw_1} = -\frac{\partial^2(R_\circ + R_1)}{\partial X_2 \partial X_3}\delta X_2 - \frac{\partial^2(R_\circ + R_1)}{\partial X_3^2}\delta X_3;$$

hence:

$$\frac{d^2\delta\psi}{dw_1^2} = \left\{k_2^2 \frac{\partial^2(R_\circ + R_1)}{\partial X_2^2} + 2k_2 k_3 \frac{\partial^2(R_\circ + R_1)}{\partial X_2 \partial X_3} + k_3^2 \frac{\partial^2(R_\circ + R_1)}{\partial X_3^2}\right\} R_1 \delta\psi.$$

So a new argument is introduced, the amplitude being free; the square of the mean motion in w_1, if existing, is equal to the coefficient of $\delta\psi$ with reversed sign; if $\delta\psi$ has been determined, δX_2, δX_3, δx_2, δx_3 follow by quadrature from the differential equations.

The variables x_2, x_3 introduce only one new argument: a linear function of w_1 with a mean motion (though necessarily not always) equal to

$$\frac{1}{k_3}\left(\frac{dx_2}{dw_1} - \text{nearest integer}\right) \text{ and to}$$

$$-\frac{1}{k_2}\left(\frac{dx_3}{dw_1} - \text{nearest integer}\right);$$

generally, the period will be rather large, though not excessively.

The solution may be approximated to more closely by taking account of higher powers of the δ-variations and including more arguments from the function R. Then, the number of independent phase-arguments is also augmented. Evidently the subject is strictly analogous to the theory of the motion of planets and satellites in celestial mechanics: from a formal point of view the development in periodic series of the coordinates in the lunar theory is illustrative.

4. *The maintenance of the periodic motion.*

If the solution of the equations of motion has been constructed, the values of the integration constants not included in additive constants in the phase-arguments must be so determined as to satisfy the conditions necessary to maintain the multiple-periodic motion, derived in sections 1 and 2.

The general relation

$$\sum_1^\infty \alpha_i \overline{J_i \frac{dw_i}{dt}} = 0,$$

though not sufficient to determine the amplitudes if the motion is not simple-periodic, is important as it shows necessary restrictions in their values. For suppose a multiple-periodic solution to exist and combine the independent phase-arguments, now functions of t; then C_i is equal to an infinite sum of terms, each being equal to a constant multiplied by the cosinus of a linear function of the time; however, as

$$2\, J_i \frac{d\,w_i}{dt} = \left(\frac{d\,C_i}{dt}\right)^2 - C_i \frac{d^2\,C_i}{dt^2}\,,$$

the mean value of $2\, J_i \frac{d\,w_i}{dt}$ is equal to the sum of the squares of these constants each multiplied by the mean motion of the corresponding argument. Hence, if only α_1 is negative, an upper limit exists for the absolute value of each product of amplitude and mean motion occurring in $C_2, C_3 \ldots$; generally as $\alpha_2, \alpha_3 \ldots$ are far larger than $-\alpha_1$, this upper limit is relatively small. Moreover, as the coefficients of the second and higher harmonics in the radial velocity largely depend on $C_2, C_3, \ldots$, the order of magnitude of these coefficients may be understood.

If only a simple-periodic solution is considered, then the relation is sufficient to determine the value of the amplitude exactly.

If the function H is restricted to one periodic term preponderant in case of a close commensurability in the ratio 1 : 2 between the frequencies in the first and second fundamental mode of vibration, the equation is reduced to the simple relation between the constant values of J_1 and J_2:

$$\alpha_1 J_1 + 2\,\alpha_2 J_2 = 0\,;$$

hence α_1 and α_2 must have opposite sign if a solution is possible. As the functional relation between J_2 and J_1 results from the construction of the periodic solution [1]), this equation determines the value of J_1, hence of the amplitude of the pulsation.

A general quantitative determination of the constants of integration involved in a multiple-periodic solution is far too intricate; hence a qualitative treatment must suffice, corresponding to the restriction of the function R to a single periodic term in the previous section.

There the integration of the differential equations has introduced two independent phase-arguments, linear functions of w_1, each connected with a constant of integration.

Identification of these arguments with two of the arguments χ_k establishes two relations for the determination of these constants of integration.

Firstly, the relation corresponding to the χ-argument contained in those parts of x_2 and x_3 that are linear functions of w_1 is to be considered. The dependent variable $n_1 t$ does not contain this χ-argument, hence the relation is simplified to the form:

$$\alpha_2 \overline{J_2 \frac{\partial\, w_2}{\partial \chi}} + \alpha_3 \overline{J_3 \frac{\partial\, w_3}{\partial \chi}} = 0\,.$$

Transformation to the variables X, x reduces this relation to the equation:

$$\alpha_2 \overline{X_2 \frac{\partial\, x_2}{\partial \chi}} + \alpha_3 \overline{X_3 \frac{\partial\, x_3}{\partial \chi}} = 0\,,$$

hence to the equation:

$$\alpha_2\, k_3\, X_2 = \alpha_3\, k_2\, X_3\,.$$

X_2 and X_3 are positive quantities to be connected by the relation

$$k_1 = k_2 \frac{\partial\,(R_0 + R_1)}{\partial\, X_2} + k_3 \frac{\partial\,(R_0 + R_1)}{\partial\, X_3}\,.$$

As α_2 and α_3 are supposed to be positive, the solution of the two equations by non-zero values of both X_2 and X_3 requires k_2 and k_3 to be either both positive and non-zero or both negative and non-zero. Then a secondary oscillation may be maintained; the period is probably large compared with the fundamental period, though not excessively large.

Secondly, the relation corresponding to the χ-argument contained in the δ-variations is to be considered. Here the mean value of $J_i \frac{\partial\, w_i}{\partial \chi}$ $(i = 2, 3)$, being equal to the mean value of $X_i \frac{\partial\, x_i}{\partial \chi}$, is equal to the mean value of $\delta\, X_i \frac{\partial\, \delta\, x_i}{\partial \chi}$, hence to the product of, respectively,

$$-\left\{k_2^{\,2} \frac{\partial^2 (R_0 + R_1)}{\partial\, X_2^{\,2}} + k_2\, k_3 \frac{\partial^2 (R_0 + R_1)}{\partial\, X_2\, \partial\, X_3}\right\} \frac{R_1^{\,2}}{2}\,,\; -\left\{k_2\, k_3 \frac{\partial^2 (R_0 + R_1)}{\partial\, X_2\, \partial\, X_3} + k_3^{\,2} \frac{\partial^2 (R_0 + R_1)}{\partial\, X_3^{\,2}}\right\} \frac{R_1^{\,2}}{2}$$

and the square of the free amplitude divided by the value of the mean motion of the argument χ. The variation in $n_1 t$ must be computed from the equation:

$$\frac{d\,\delta(n_1 t)}{d\,w_1} = \frac{\partial^2 (Q_0 + Q_1)}{\partial\, z\, \partial J_2}\, \delta\, X_2 + \frac{\partial^2 (Q_0 + Q_1)}{\partial\, z\, \partial J_3}\, \partial\, X_3\,,$$

Q_0 and Q_1 being those terms in the function Q that correspond to the restriction of the function R to R_0, R_1. Hence the mean value of $J_i \frac{\partial (n_1 t)}{\partial \chi}$ $(i = 2, 3)$ is equal to the product of, respectively,

$$\left\{k_2^{\,2} \frac{\partial^2 (Q_0 + Q_1)}{\partial\, z\, \partial J_2} + k_2\, k_3 \frac{\partial^2 (Q_0 + Q_1)}{\partial\, z\, \partial J_3}\right\} \frac{R_1^{\,2}}{2}\,,\; \left\{k_2\, k_3 \frac{\partial^2 (Q_0 + Q_1)}{\partial\, z\, \partial J_2} + k_3^{\,2} \frac{\partial^2 (Q_0 + Q_1)}{\partial\, z\, \partial J_3}\right\} \frac{R_1^{\,2}}{2}$$

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and the square of the amplitude divided by the mean motion of the argument χ.

The equation of condition for maintenance of the pulsation corresponding to the argument χ now being considered, reduced to its principal terms is:

$$\alpha_2 \left(\overline{J_2 \frac{\partial w_2}{\partial \chi}} + m_2 \overline{J_2 \frac{\partial (n_1 t)}{\partial \chi}} \right) + \alpha_3 \left(\overline{J_3 \frac{\partial w_3}{\partial \chi}} + m_3 \overline{J_3 \frac{\partial (n_1 t)}{\partial \chi}} \right) = 0 ;$$

m_i are integers approximating to the influence of the coefficients $\frac{d w_i}{d w_1}$. Substitution of the computed mean values transforms this equation into the equation of condition:

$$\alpha_2 \left\{ k_2^2 \frac{\partial^2 (R_0 + R_1)}{\partial X_2^2} + k_2 k_3 \frac{\partial^2 (R_0 + R_1)}{\partial X_2 \partial X_3} \right\} + \alpha_3 \left\{ k_2 k_3 \frac{\partial^2 (R_0 + R_1)}{\partial X_2 \partial X_3} + k_3^2 \frac{\partial^2 (R_0 + R_1)}{\partial X_3^2} \right\} =$$

$$m_2 \alpha_2 \left\{ k_2^2 \frac{\partial^2 (Q_0 + Q_1)}{\partial z \partial J_2} + k_2 k_3 \frac{\partial^2 (Q_0 + Q_1)}{\partial z \partial J_3} \right\} + m_3 \alpha_3 \left\{ k_2 k_3 \frac{\partial^2 (Q_0 + Q_1)}{\partial z \partial J_2} + k_3^2 \frac{\partial^2 (Q_0 + Q_1)}{\partial z \partial J_3} \right\} .$$

As R_1 generally is proportional to powers of $\sqrt{X_2}$, $\sqrt{X_3}$ the derivatives $\frac{\partial^2 R_1}{\partial X_2^2}, \frac{\partial^2 R_1}{\partial X_2 \partial X_3}, \frac{\partial^2 R_1}{\partial X_3^2}$ may be estimated to be of the order of magnitude of R_1 divided by X_2^2, $X_2 X_3$, X_3^2 respectively. Furthermore as α_3 predominates, R_1 must be as regards order of magnitude comparable with the differential coefficients of R_0 of second order with regard to X_2, X_3, multiplied by X_3^2. Hence the square of the mean motion of the argument as computed in the preceding section must be of the order of magnitude of X_3 multiplied by the said differential coefficients. As the value of X_3 is limited by the general upper-limit condition discussed in the first part of this section, the period must be excessively long.

This analysis of the state of motion and its maintenance corresponding to the restricted function R might also have been used, slightly changed, to gain some preliminary insight in the construction of the simple-periodic solution already carried out more generally in the former part of this section.

It is to be born in mind that a quantitative determination of the constants of integration involved in the maintained multiple-periodic solution involves a far more extensive and intricate analysis. However, the existence of the upper-limit condition may replace the analysis as far as concerns a comparison with observation.

5. *Some special developments.*

It is useful to apply the preceding general analysis to some extent to the special consideration of a function H composed only of some terms of principal importance if the values of n_1, n_2, n_3 are nearly commensurable in the ratio 1 : 2 : 3. The coefficients of the goniometric terms have been computed in a former paper [1]).

The function H is restricted to the periodic terms in $2w_1 - w_2$ and $3w_1 - w_3$, hence:

$$H = n_1 J_1 + n_2 J_2 + n_3 J_3 + k_{12} J_1 \sqrt{J_2} \cos(2 w_1 - w_2) + k_{13} J_1^{3/2} \sqrt{J_3} \cos(3 w_1 - w_3) .$$

The formation of the equations of motion with w_1 as independent variable requires the solution from this equation of J_1 as a function of J_2, J_3, w_1, w_2, w_3 and z, the functional relation denoted by the symbol Q. Hence, to a sufficient degree of approximation, the function Q is determined by the relation:

$$= z - \frac{n_2}{n_1} J_2 - \frac{n_3}{n_1} J_3 - \frac{k_{12}}{n_1} \left\{ z - \frac{n_2}{n_1} J_2 - \frac{n_3}{n_1} J_3 \right\} \sqrt{J_2} \cos(2w_1 - w_2) - \frac{k_{13}}{n_1} z^{3/2} \left\{ 1 - \frac{3}{2} \frac{n_2}{n_1} \frac{J_2}{z} - \frac{3}{2} \frac{n_3}{n_1} \frac{J_3}{z} \right\} \sqrt{J_3} \cos(3w_1 - w_3) ;$$

where necessary the approximation may easily be continued.

Hence the simple-periodic solution is given by constant values of J_2 and J_3, to be denoted by A_2 and A_3 and values of $2w_1 - w_2$ and $3w_1 - w_3$ equal to either 0 or π, the choice being so made as to ensure the positive sign of $\sqrt{J_2}$ and $\sqrt{J_3}$; suppose 0 to be the required value. These constant values result from the solution of the equations:

$$2 + \frac{\partial Q}{\partial J_2} = 0 , \quad 3 + \frac{\partial Q}{\partial J_3} = 0 .$$

Then this periodic solution is to be used in introducing the new variables X_2, X_3, x_2, x_3 by the equations of transformation:

$$\sqrt{J_2} \cos w_2 = \sqrt{A_2} \cos 2 w_1 + \sqrt{X_2} \cos x_2 , \quad \sqrt{J_3} \cos w_3 = \sqrt{A_3} \cos 3 w_1 + \sqrt{X_3} \cos x_3 ,$$

$$\sqrt{J_2} \sin w_2 = \sqrt{A_2} \sin 2 w_1 + \sqrt{X_2} \sin x_2 , \quad \sqrt{J_3} \sin w_3 = \sqrt{A_3} \sin 3 w_1 + \sqrt{X_3} \sin x_3 .$$

1) *B.A.N.* No. 359.