On the extension of the theory of adiabatic Cepheid pulsation

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BULLETIN OF THE ASTRONOMICAL INSTITUTES OF THE NETHERLANDS.

1936 February 7. Volume VII. No. 276.

COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On the extension of the theory of adiabatic Cepheid pulsation, by Miss *H. A. Kluyver.*

1. In the theory of adiabatically pulsating stars, developed in order to explain Cepheid variability, the usual method of investigating second-order terms breaks down in the particular case where two of the characteristic frequencies are approximately commensurable.

Dr WOLTJER has shown [1]) how the theory of periodic solutions from celestial mechanics may be applied here, and the present paper contains a solution of EDDINGTON's equations, obtained with the aid of this method. Further it is shown that the observed changes in radial velocity of the cluster-type variable RR Lyrae can be interpreted as due to approximate commensurability between the first and third characteristic frequencies.

2. The equation for the displacement is given by the hydrodynamical equation, combined with the first law of thermodynamics in the adiabatic form and the equation that expresses conservation of mass.

The hydrodynamical equation is:

$$\frac{dP}{dr} = - g\rho - \rho \frac{d^2(r_\circ\zeta)}{dt^2},$$

where: r = distance from star's centre,
P = total pressure = gas pressure + radiation pressure,
g = acceleration of gravity,
ρ = density,
$\zeta = \delta r_\circ / r_\circ$.

Quantities with index zero are to be given the equilibrium values for a star with polytropic index equal to 3.

The conservation of mass leads to the equation:

$$\frac{\rho}{\rho_\circ} = (1+\zeta)^{-2}\left(1+\zeta+r_\circ\frac{d\zeta}{dr_\circ}\right)^{-1}.$$

[1]) *M. N.* **95**, p. 260, 1935.

For adiabatic oscillations the first law of thermodynamics gives:

$$\frac{P}{P_\circ} = \left(\frac{\rho}{\rho_\circ}\right)^\gamma,$$

where γ is the ratio of specific heats for matter and radiation taken together. Thus $P/P_\circ$ can be expressed in ζ and $d\zeta/dr_\circ$.

The hydrodynamical equation may be written:

$$\frac{dP}{r_\circ^2\rho_\circ dr_\circ} = -\frac{g_\circ r_\circ^2}{r^4} - \frac{r_\circ}{r^2}\frac{d^2\zeta}{dt^2},$$

which gives:

$$\frac{d^2\zeta}{dt^2} = -\frac{r^2}{r_\circ^3\rho_\circ}\frac{dP}{dr_\circ} - \frac{g_\circ r_\circ}{r^2} \qquad \text{(I)}.$$

All quantities occurring in equation (I) can be developed in terms of ζ; the first-order terms give:

$$\frac{d^2\zeta}{dt^2} = -\frac{P_\circ}{r_\circ\rho_\circ}\frac{d(P/P_\circ)}{dr_\circ} + \frac{g_\circ(P-P_\circ)}{r_\circ P_\circ} + \frac{4g_\circ\zeta}{r_\circ}.$$

With $\zeta = s\cos nt$ and, as new independent variable the distance from the centre, z, measured in EMDEN's units, this equation becomes:

$$\frac{d^2s}{dz^2} + \frac{4-\mu}{z}\frac{ds}{dz} + \left\{\frac{\rho_c n^2}{P_c\gamma}\left(\frac{R}{Z}\right)^2\frac{1}{u} - (3-4/\gamma)\frac{\mu}{z^2}\right\}s = 0;$$

R and Z are the values of $r_\circ$ and z at the outer boundary of the star, $\mu = \frac{g_\circ\rho_\circ r_\circ}{P_\circ} = -4\frac{z}{u}\frac{du}{dz}$, ρ_c and P_c are the central values of $\rho_\circ$ and $P_\circ$, and $u = \left(\frac{\rho_\circ}{\rho_c}\right)^{1/3}$.

With:

$$\frac{\rho_c n^2}{P_c\gamma}\left(\frac{R}{Z}\right)^2 = \omega^2,$$

this is EDDINGTON's well-known equation, which may be written, after multiplication with z^4u^4, in the form:

$$\frac{d}{dz}\left\{z^4u^4\frac{ds}{dz}\right\} + z^4u^4\left\{\frac{\omega^2}{u} - (3-4/\gamma)\frac{\mu}{z^2}\right\}s = 0.$$

An infinite number of values for ω^2 exist, such that for any of these the equation has a solution s_ω

that satisfies both boundary conditions of being finite at the centre and at the outer boundary. The functions s_ω form a complete set of orthogonal functions; they may be normalized.

3. Equation (I) may be written:

$$\frac{d^2\zeta}{dt^2} = -\left(\frac{Z}{R}\right)^2 \left\{ \frac{(1+\zeta)^2}{z\rho_0} \frac{\partial P}{\partial z} + \frac{\mu P_0}{z^2(1+\zeta)^2 \rho_0} \right\}.$$

The solution ζ can be developed in the characteristic functions, in the form:

$$\zeta = \sum_\omega C_\omega(t)\, s_\omega(z),$$

where the coefficients C_ω depend on the time only. Differential equations for the coefficients C_ω, including terms to an arbitrary degree in C_ω, are derived by developing the right-hand member also in terms of the functions s_ω and equating the coefficients of s_ω in both members. The coefficient of s_ω in the development of the right-hand member is found by multiplying it with $z^4u^3s_\omega\, dz$ and integrating from 0 to Z; it is thus equal to:

$$-\frac{1}{\rho_c}\left(\frac{Z}{R}\right)^2 \left\{ \int_0^Z (1+\zeta)^2 z^3 \frac{\partial\zeta}{\partial C_\omega}\frac{\partial P}{\partial z}dz + \int_0^Z \frac{\mu P_0 z^2}{(1+\zeta)^2}\frac{\partial\zeta}{\partial C_\omega}dz \right\},$$

since $\rho_0 = \rho_c u^3$ and $s_\omega = \partial\zeta/\partial C_\omega$.

The first integral, I_1, may be transformed by partial integration, the integrated terms disappearing at the boundaries, into:

$$I_1 = -\int_0^Z P\frac{\partial}{\partial z}\left\{(1+\zeta)^2 z^3 \frac{\partial\zeta}{\partial C_\omega}\right\} dz =$$

$$= -\int_0^Z P z^2 \left\{ 2(1+\zeta)z\frac{\partial\zeta}{\partial z}\frac{\partial\zeta}{\partial C_\omega} + 3(1+\zeta)^2\frac{\partial\zeta}{\partial C_\omega} + (1+\zeta)^2 z \frac{\partial^2\zeta}{\partial z\,\partial C_\omega}\right\} dz = -\int_0^Z P z^2 \frac{\partial}{\partial C_\omega}\left(\frac{\rho_0}{\rho}\right) dz.$$

Since: $$\frac{\partial}{\partial C_\omega}\left(\frac{P}{\rho}\right) = P\frac{\partial\, {}^1/\rho}{\partial C_\omega} + \frac{1}{\rho}\frac{\partial P}{\partial C_\omega} = P\frac{\partial\, {}^1/\rho}{\partial C_\omega} + \frac{\gamma P}{\rho^2}\frac{\partial \rho}{\partial C_\omega} = -\left(\gamma - 1\right) P\frac{\partial\, {}^1/\rho}{\partial C_\omega},$$

this last integral is equal to:

$$+\frac{\partial}{\partial C_\omega}\int_0^Z \frac{P}{(\gamma-1)\rho} z^2 \rho_0\, dz.$$

As to the second integral, it is evident that:

$$I_2 = \int_0^Z \frac{\mu z^2 P_0}{(1+\zeta)^2}\frac{\partial\zeta}{\partial C_\omega}dz = -\frac{\partial}{\partial C_\omega}\int_0^Z \frac{\mu z^2 P_0}{1+\zeta}dz.$$

Hence the differential equation for C_ω is found to be:

$$\frac{d^2C_\omega}{dt^2} = -\frac{1}{\rho_c}\left(\frac{Z}{R}\right)^2 \frac{\partial}{\partial C_\omega}\left\{ \int_0^Z \frac{P}{(\gamma-1)\rho} z^2\rho_0\, dz - \int_0^Z \frac{\mu z^2 P_0}{1+\zeta}dz \right\} \qquad \text{(II).}$$

The first integral corresponds to the internal energy, the second one to the gravitational energy. In the right-hand member the terms of different degree in C_ω are treated separately, and therefore the integrands are developed in powers of ζ.

Since $\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$ and $P_0 = P_c\left(\frac{\rho_0}{\rho_c}\right)^{4/3} = P_c u^4$,

the first integrand may be written:

$$\frac{P_c}{\gamma-1} z^2 u^4 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \frac{P_c}{\gamma-1} z^2 u^4 \left(1+\zeta\right)^{-2(\gamma-1)}\left(1+\zeta+z\frac{\partial\zeta}{\partial z}\right)^{-(\gamma-1)}.$$

The second integrand becomes:

$$P_c\frac{\mu z^2 u^4}{1+\zeta}.$$

Thus: $$\frac{d^2C_\omega}{dt^2} = -\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\frac{\partial}{\partial C_\omega}\left\{\int_0^Z \frac{z^2u^4}{\gamma-1}\left(1+\zeta\right)^{-2(\gamma-1)}\left(1+\zeta+z\frac{\partial\zeta}{\partial z}\right)^{-(\gamma-1)} dz - \int_0^Z \frac{\mu z^2 u^4}{1+\zeta}dz\right\}.$$

The terms of order zero disappear with the differentiation. Those of the first order are:

$$-\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\left\{-3\int_0^Z z^2 u^4 \zeta\, dz - \int_0^Z z^3 u^4 \frac{\partial \zeta}{\partial z} dz + \int_0^Z \mu\, z^2 u^4 \zeta\, dz\right\}.$$

They are seen to be identically zero, since by partial integration, the integrated parts disappearing, the second integral may be shown to cancel the two others:

$$-\int_0^Z z^3 u^4 \frac{\partial \zeta}{\partial z} dz = +3\int_0^Z z^2 u^4 \zeta\, dz - \int_0^Z \mu\, z^2 u^4 \zeta\, dz.$$

The terms of the second order are:

$$\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\left[\int_0^Z\left\{\left(3-\frac{9}{2}\gamma\right)\zeta^2 + (2-3\gamma)\,\zeta z\frac{\partial \zeta}{\partial z} - \frac{1}{2}\gamma z^2\left(\frac{\partial \zeta}{\partial z}\right)^2\right\} z^2 u^4\, dz + \int_0^Z \mu\, z^2 u^4 \zeta^2\, dz\right] =$$

$$= \frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\left[\int_0^Z\left\{\left(1-\frac{3}{2}\gamma\right)u^4\frac{\partial}{\partial z}(z^3\zeta^2) - \frac{1}{2}\gamma z^4 u^4\left(\frac{\partial \zeta}{\partial z}\right)^2\right\} dz + \int_0^Z \mu\, z^2 u^4 \zeta^2\, dz\right].$$

By partial integration the first term in the first integral is transformed into:

$$\left(1-\frac{3}{2}\gamma\right)\int_0^Z \mu\, z^2 u^4 \zeta^2\, dz,$$

and the second term into:

$$+\frac{1}{2}\gamma\int_0^Z \zeta\frac{\partial}{\partial z}\left(z^4 u^4 \frac{\partial \zeta}{\partial z}\right) dz,$$

so that all terms taken together give:

$$\frac{1}{2}\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\gamma\int_0^Z \zeta\left\{\frac{\partial}{\partial z}\left(z^4 u^4 \frac{\partial \zeta}{\partial z}\right) - (3-4/\gamma)\,\mu\, z^2 u^4 \zeta\right\} dz.$$

After the substitution $\zeta = \Sigma C_\omega s_\omega$ the form in brackets may, with the aid of the differential equation for s_ω, be written:

$$-\sum \omega^2 C_\omega s_\omega z^4 u^3.$$

On account of the orthogonality of the functions s_ω the only terms left after the integration are:

$$-\frac{1}{2}\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\gamma\sum \omega^2 C^2_\omega = -\sum\frac{1}{2}n^2 C^2_\omega.$$

Thus the differential equation for C_ω, to the first order in C_ω, becomes:

$$\frac{d^2 C_\omega}{dt^2} = -n^2 C_\omega.$$

4. In the case of approximate commensurability the second-order terms in the differential equation become critical. These are found from:

$$\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\frac{\partial}{\partial C_\omega}\left[\int_0^Z z^2 u^4\left\{\left(1-\frac{9}{2}\gamma+\frac{9}{2}\gamma^2\right)\zeta^3 + \left(1-\frac{9}{2}\gamma+\frac{9}{2}\gamma^2\right)\zeta^2 z\frac{\partial \zeta}{\partial z} + \right.\right.$$

$$\left.\left. + \frac{1}{2}\gamma\,(3\gamma-1)\,\zeta z^2\left(\frac{\partial \zeta}{\partial z}\right)^2 + \frac{1}{6}\gamma\,(\gamma+1)\,z^3\left(\frac{\partial \zeta}{\partial z}\right)^3\right\} dz - \int_0^Z \mu\, z^2 u^4 \zeta^3\, dz\right],$$

which expression may, by partial integration of the second term in the first integral, be transformed into:

$$\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\frac{\partial}{\partial C_\omega}\int_0^Z\left\{\left(\frac{3}{2}\gamma^2-\frac{3}{2}\gamma-\frac{2}{3}\right)\mu\,\zeta^3 + \frac{1}{2}\gamma\,(3\gamma-1)\,\zeta z^2\left(\frac{\partial \zeta}{\partial z}\right)^2 + \frac{1}{6}\gamma\,(\gamma+1)\,z^3\left(\frac{\partial \zeta}{\partial z}\right)^3\right\} z^2 u^4\, dz.$$

The function ζ contains all functions s_ω, but since this investigation is concerned with the effect of commensurability, only the first characteristic oscillation and the one which has a frequency equal to twice the first characteristic frequency, have been included.

At first it had been expected that for $3-4/\gamma$ equal to 0·4 the second characteristic frequency would be about twice the first one; however, it turned out that for this value of the parameter the *third* and first characteristic frequencies are approximately commensurable, whereas for the second and first ones this is the case for $3-4/\gamma$ somewhat smaller than 0·2. It is possible that for values of the parameter between 0·4 and the maximum 0·6 the fourth and first characteristic frequencies are commensurable. The value 0·2 is less probable on account of the hydrogen content of the stars and the case where $3-4/\gamma = 0{\cdot}4$ has been investigated here.

The characteristic values of ω^2 and the corresponding functions s_ω have been found according to the method formerly used [1]), which has been slightly refined in the following way. At $z = 5{\cdot}0$ the quotient of ds/dz and s is determined both for the solution starting at the centre and for that from the outer boundary; the characteristic frequency is the one for which these two quotients are equal. The final values are $\omega^2 = {\cdot}10391$ for the first and $\omega'^2 = {\cdot}39179$ for the third characteristic oscillation [2]). Details of the successive approximations are shown in Table 1. The functions u and μ have been taken from the *B. A. Mathematical Tables, vol. II*, the solutions for $\omega^2 = {\cdot}110$ and $\omega'^2 = {\cdot}466$ for z from 0 to 5·0, from the paper by J. A. EDGAR.

TABLE 1.

First characteristic oscillation				Third characteristic oscillation			
ω^2	$\frac{1}{s}\frac{ds}{dz}$ (inside)	$\frac{1}{s}\frac{ds}{dz}$ (outside)	difference	ω'^2	$\frac{1}{s}\frac{ds}{dz}$ (inside)	$\frac{1}{s}\frac{ds}{dz}$ (outside)	difference
·11000	+ ·29150	+ ·52187	— ·23037	·46600	— ·9645	+ 1·5556	— 2·5201
·10400	+ ·47383	+ ·47730	— ·00347	·42000	— ·1065	+ ·7514	— ·8579
				·39091	+ ·2944	+ ·2685	+ ·0259
·10391	+ ·47627	+ ·47668	— ·00041	·39179	+ ·2834	+ ·2820	+ ·0014

All solutions of the equation for s must, at the centre and at the outer boundary, be started with the aid of a series, on account of singularity of the coefficients. In the outer part of the star the series for the first characteristic oscillation can be used as far inward as $z = 5{\cdot}0$.

Numerical data about the trial solutions and the characteristic functions s_ω are given in the following tables; all start with value unity. Table 2 contains the second trial solution for the first characteristic vibration, the first one having been taken from EDGAR's paper [3]). Corresponding data for the third characteristic vibration may be found in Table 4. The solutions for the final values ω^2 and ω'^2, one for the inner and one for the outer part of the star, are given in Table 5. They have been computed by extrapolation and interpolation from the trial solutions; for the first characteristic oscillation the series has been used in the outer part of the star. The coefficients of this series, together with those from which they have been extrapolated, are shown in Table 3. If the two parts of each final solution are combined into one, the values at the outer boundary become, for unit central amplitude, 9·0499 and 114·34; for the normalized functions these boundary values are 1·47042 and 22·520.

TABLE 2.
Solution for $\omega^2 = {\cdot}10400$.

z	s	$\frac{ds}{dz}$	$\frac{d^2s}{dz^2}$
0·0	+ 1·00000	·00000	+ ·0859
0·2	1·00172	+ ·01722	·0866
0·4	1·00691	·03475	·0889
0·6	1·01566	·05290	·0928
0·8	1·02813	·07198	·0983
1·0	1·04454	·09231	·1054
1·2	1·06516	·11425	·1143
1·4	1·09037	·13814	·1250
1·6	1·12058	·16439	·1378
1·8	1·15631	·19340	·1527
2·0	1·19815	·22561	·1699
2·2	1·24680	·26152	·1897
2·4	1·30304	·30167	·2123
2·6	1·36779	·34665	·2380
2·8	1·44207	·39710	·2671
3·0	1·52704	·45376	·3001
3·2	1·62403	·51742	·3373
3·4	1·73454	·58898	·3792
3·6	1·86022	·66945	·4264
3·8	2·00298	·75991	·4793
4·0	2·16493	·86160	·5387
4·2	2·34846	·97587	·6053
4·4	2·55622	1·10422	·6796
4·6	2·79119	1·24825	·7622
4·8	3·05668	1·40970	·8538
5·0	+ 3·35635	+ 1·59034	+ ·9539

1) *B.A.N.* **7**, p. 265, 1935.

2) This result is different from the one found by J. A. EDGAR, *M. N.* **93**, p. 422, 1933, viz. $\omega^2 = {\cdot}466$ for the *second* characteristic vibration; however, his method, where the outer part of the star can hardly be taken into account, is not so suitable for a determination of the higher characteristic oscillations.

3) l.c., Table III, p. 428.

Table 3.

Coefficients in the series $s = \sum_i k_i \left(1 - \frac{z}{Z}\right)^i$, near the outer boundary.

	$\omega^2 =$ ·11000	$\omega^2 =$ ·10400	$\omega^2 =$ ·10391
i	k_i	k_i	k_i
0	+ 1·0000	+ 1·0000	+ 1·0000
1	— 4·0700	— 3·8262	— 3·8225
2	+ 8·7340	+ 7·8664	+ 7·8531
3	— 12·3991	— 10·6953	— 10·6692
4	+ 13·3154	+ 11·0670	+ 11·0326
5	— 11·8120	— 9·5088	— 9·4736
6	+ 9·198	+ 7·223	+ 7·193
7	— 6·538	— 5·025	— 5·002
8	+ 4·305	+ 3·250	+ 3·234
9	— 2·658	— 1·966	— 1·956

Table 4.

Solutions for $\omega'^2 =$ ·42000 and $\omega'^2 =$ ·39091.

	$\omega'^2 =$ ·42000			$\omega'^2 =$ ·39091		
z	s	$\frac{ds}{dz}$	$\frac{d^2s}{dz^2}$	s	$\frac{ds}{dz}$	$\frac{d^2s}{dz^2}$
0·0	+ 1·00000	·0000	+ ·0227	+ 1·00000	·0000	+ ·0285
0·2	1·00045	+ ·0044	·0208	1·00056	+ ·0056	·0268
0·4	1·00171	·0080	·0149	1·00219	·0105	·0217
0·6	1·00356	·0101	+ ·0045	1·00467	·0140	+ ·0125
0·8	1·00557	·0095	— ·0113	1·00763	·0152	— ·0016
1·0	1·00710	+ ·0051	·0339	1·01051	·0129	·0218
1·2	1·00725	— ·0046	·0649	1·01249	+ ·0059	·0497
1·4	1·00477	·0216	·1065	1·01245	— ·0076	·0873
1·6	·99798	·0481	·1609	1·00889	·0298	·1370
1·8	·98471	·0870	·2308	·99980	·0633	·2014
2·0	·96214	·1416	·3189	·98259	·1115	·2834
2·2	·92674	·2159	·4275	·95397	·1781	·3859
2·4	·87418	·3141	·5582	·90985	·2674	·5113
2·6	·79921	·4407	·7111	·84519	·3842	·6610
2·8	·69574	·5998	·8834	·75400	·5334	·8345
3·0	·55688	·7949	1·0681	·62937	·7194	1·0280
3·2	·37530	1·0270	1·2510	·46359	·9453	1·2322
3·4	+ ·14377	1·2936	1·4075	+ ·24853	1·2118	1·4293
3·6	— ·14384	1·5856	1·4979	— ·02357	1·5145	1·5885
3·8	·49095	1·8842	1·4604	·35892	1·8414	1·6599
4·0	·89575	2·1552	1·2030	·76012	2·1675	1·5656
4·2	1·34749	2·3420	— ·5919	1·22301	2·4488	1·1873
4·4	1·82112	2·3560	+ ·5617	1·73187	2·6120	— ·3489
4·6	2·26962	2·0640	2·5179	2·25232	2·5408	+ 1·2072
4·8	2·61359	— 1·2725	5·6221	2·72087	2·0568	3·8531
5·0	— 2·72739	+ ·2905	+ 10·3134	— 3·02993	— ·8919	+ 8·1208
5·0	— ·024569	— ·01846	+ ·0488	— ·026476	— ·00711	+ ·0724
5·2	·026962	— ·00365	·1037	·026103	+ ·01278	·1307
5·4	·025101	+ ·02515	·1907	·020402	·04716	·2189
5·6	— ·015454	·07579	·3250	— ·005832	·10320	·3499
5·8	+ ·007422	·15973	·5280	+ ·022966	·19113	·5412
6·0	·051744	·29353	·8296	·073690	·32528	·8169
6·2	·129716	·50090	1·2719	·157482	·52574	1·2106
6·4	·259224	·81553	1·9134	·290240	·82038	1·7675
6·6	·466206	1·28498	2·8359	·494450	1·24777	2·5499
6·8	·787930	1·97599	4·1509	·801687	1·86092	3·6415
7·0	1·277504	2·98171	6·0132	1·255984	2·73247	5·1559
7·2	+ 2·010123	+ 4·43172	+ 8·6348	+ 1·918431	+ 3·96159	+ 7·2468

TABLE 5.
Final solutions for the first and third characteristic oscillations.

z	$\omega^2 =$ ·10391 s	$\frac{ds}{dz}$	$\omega'^2 =$ ·39179 s	$\frac{ds}{dz}$
0·0	+ 1·00000	·00000	+ 1·00000	·0000
0·2	1·00172	+ ·01722	1·00056	+ ·0056
0·4	1·00691	·03476	1·00218	·0104
0·6	1·01566	·05291	1·00464	·0139
0·8	1·02813	·07198	1·00757	·0150
1·0	1·04455	·09233	1·01041	·0127
1·2	1·06517	·11427	1·01233	+ ·0057
1·4	1·09039	·13818	1·01222	— ·0080
1·6	1·12061	·16445	1·00856	·0304
1·8	1·15635	·19348	·99934	·0640
2·0	1·19821	·22572	·98197	·1124
2·2	1·24687	·26167	·95315	·1792
2·4	1·30315	·30187	·90877	·2688
2·6	1·36795	·34692	·84380	·3859
2·8	1·44229	·39747	·75224	·5354
3·0	1·52734	·45426	·62717	·7217
3·2	1·62445	·51809	·46092	·9478
3·4	1·73512	·58988	+ ·24536	1·2143
3·6	1·86100	·67066	— ·02721	1·5167
3·8	2·00405	·76154	·36292	1·8427
4·0	2·16638	·86383	·76423	2·1671
4·2	2·35044	·97891	1·22678	2·4456
4·4	2·55891	1·10835	1·73457	2·6042
4·6	2·79471	1·25386	2·25284	2·5264
4·8	3·06121	1·41764	2·71762	2·0330
5·0	+ 3·36269	+ 1·60155	— 3·02076	— ·8561
5·0	+ ·37157	+ ·17712	— ·026418	— ·00745
5·2	·40925	·20008	·026129	+ ·01228
5·4	·45181	·22587	·020544	·04649
5·6	·49982	·25488	— ·006124	·10237
5·8	·55399	·28752	+ ·022495	·19018
6·0	·61509	·32426	·073025	·32432
6·2	·68400	·36557	·156641	·52499
6·4	·76168	·41204	·289300	·82023
6·6	·84922	·46434	·493594	1·24890
6·8	·94784	·52315	·801270	1·86441
7·0	1·05897	·58936	1·256636	2·74002
7·2	+ 1·18414	+ ·66387	+ 1·921209	+ 3·97583

The term needed afterwards is the one proportional to $C^2_\omega C_{\omega'}$; its coefficient is:

$$-\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\left\{\left(2+\frac{9}{2}\gamma-\frac{9}{2}\gamma^2\right)\int_0^Z s^2_\omega\, s_{\omega'}\,\mu z^2 u^4\, dz-\frac{1}{2}\,\gamma\,(3\gamma-1)\int_0^Z\left(\frac{ds_\omega}{dz}\right)^2 s_{\omega'}\, z^4 u^4\, dz-\right.$$

$$\left.-\gamma\,(3\gamma-1)\int_0^Z s_\omega\,\frac{ds_\omega}{dz}\,\frac{ds_{\omega'}}{dz}\,z^4 u^4\, dz-\frac{1}{2}\,\gamma\,(\gamma+1)\int_0^Z\left(\frac{ds_\omega}{dz}\right)^2\frac{ds_{\omega'}}{dz}\,z^5 u^4\, dz\right\}.$$

Because the possibility of this general treatment had not been realized from the beginning, the coefficient had been derived in a more complicated form. In order to express the coefficient as found above in actually computed integrals, it may be transformed, by partial integration of the last integral and application of the differential equation for s_ω, into:

$$-\frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2\gamma\left\{\left(\frac{2}{\gamma}+\frac{9}{2}-\frac{9}{2}\gamma\right)\int_0^Z s_\omega^2\, s_{\omega'}\,\mu\, z^2 u^4\, dz-(\gamma+1)\,\omega^2\int_0^Z s_\omega\,\frac{ds_\omega}{dz}\,s_{\omega'}\,z^5 u^3\, dz+\right.$$

$$+ (\gamma + 1)\left(3 - \frac{4}{\gamma}\right)\int_0^Z s_\omega \frac{ds_\omega}{dz} s_\omega' \mu z^3 u^4 dz - (3\gamma + 1)\int_0^Z \left(\frac{ds_\omega}{dz}\right)^2 s_\omega' z^4 u^4 dz +$$

$$+ \frac{1}{2}(\gamma + 1)\int_0^Z \left(\frac{ds_\omega}{dz}\right)^2 s_\omega' \mu z^4 u^4 dz - (3\gamma - 1)\int_0^Z s_\omega \frac{ds_\omega}{dz}\frac{ds_\omega'}{dz} z^4 u^4 dz \Big\}.$$

The six integrands are given in Table 6, the integrals being shown in the last line. For the integration the interval 0·2 has been used from 0 to 6·0, and from 6·0 to Z the interval 0·1.

TABLE 6.

z	$s^2_\omega s_\omega' \mu z^2 u^4$	$s_\omega \frac{ds_\omega}{dz} s_\omega' z^5 u^3$	$s_\omega \frac{ds_\omega}{dz} s_\omega' \mu z^3 u^4$	$\left(\frac{ds_\omega}{dz}\right)^2 s_\omega' z^4 u^4$	$\left(\frac{ds_\omega}{dz}\right)^2 s_\omega' \mu z^4 u^4$	$s_\omega \frac{ds_\omega}{dz}\frac{ds_\omega'}{dz} z^4 u^4$
0·0	·0000	·000	·0000	·00000	·000	·0000
0·2	+ ·0021	·000	·0000	·00000	·000	·0000
0·4	·0306	·000	+ ·0004	+ ·00003	·000	·0000
0·6	·1352	+ ·004	·0042	·00029	·000	+ ·0001
0·8	·3562	·018	·0200	·00142	+ ·001	·0003
1·0	·6950	·061	·0614	·00460	·005	·0007
1·2	1·1110	·159	·1430	·01137	·018	+ ·0006
1·4	1·5392	·343	·2731	·02318	·048	− ·0014
1·6	1·9143	·645	·4495	·04088	·106	·0084
1·8	2·1884	1·088	·6591	·06438	·198	·0246
2·0	2·3358	1·683	·8800	·09238	·332	·0561
2·2	2·3514	2·412	1·0857	·12238	·501	·1096
2·4	2·2441	3·226	1·2476	·15068	·694	·1924
2·6	2·0290	4·029	1·3379	·17257	·882	·3112
2·8	1·7244	4·665	1·3307	·18253	1·027	·4714
3·0	1·3481	4·902	1·2028	·17457	1·073	·6753
3·2	·9169	4·424	·9357	·14281	·955	·9206
3·4	+ ·4468	+ 2·825	+ ·5165	+ ·08235	+ ·597	1·1988
3·6	− ·0449	− ·368	− ·0583	− ·00964	− ·076	1·4912
3·8	·5389	5·650	·7782	·13241	1·124	1·7692
4·0	1·0114	13·424	1·6132	·28006	2·573	1·9916
4·2	1·4331	23·828	2·5069	·44003	4·385	2·1062
4·4	1·7689	36·467	3·3711	·59243	6·425	2·0535
4·6	1·9785	50·086	4·0832	·71078	8·427	1·7767
4·8	2·0209	62·256	4·4923	·76570	9·986	1·2369
5·0	1·8624	69·219	4·4370	·73098	10·570	− ·4338
5·2	1·4862	66·113	3·7781	·59290	9·605	+ ·5700
5·4	·9096	48·024	2·4554	·36042	6·629	1·6315
5·6	− ·2011	− 12·510	− ·5742	− ·07717	− 1·640	2·5297
5·7	+ ·1656	+ 11·158	+ ·4859	+ ·06190	+ 1·426	2·8372
5·8	·5124	37·308	1·5424	·18471	4·643	3·0088
5·9	·8149	64·033	2·5157	·28074	7·766	3·0241
6·0	1·0483	88·744	3·3157	·34105	10·488	2·8730
6·1	1·1895	108·333	3·8523	·36040	12·475	2·5639
6·2	1·2219	119·550	4·0491	·33893	13·417	2·1254
6·3	1·1395	119·588	3·8607	·28302	13·081	1·6076
6·4	·9521	107·040	3·2963	·20554	11·413	1·0772
6·5	·6900	82·992	2·4396	·12409	8·626	·6081
6·6	·4049	52·044	1·4613	·05675	5·273	·2626
6·7	·1637	22·457	·6028	·01583	2·219	·0697
6·8	+ ·0268	+ 3·917	+ ·1006	+ ·00132	+ ·377	+ ·0056
6·9	·0000	·000	·0000	·00000	·000	·0000
7·0	− ·0598	− 9·898	− ·2330	+ ·00339	− ·907	+ ·0133
7·1	·6155	108·229	2·4412	·07130	9·682	·2708
7·2	− 2·7427	− 511·881	− 11·0713	+ ·49110	− 44·691	+ 1·8128
integrals	+ 2·4562	+ 10·292	− ·8453	− ·4595	− 1·872	− ·5486

With these numerical values the coefficient of $C^2{}_\omega C_\omega'$ is found to be equal to

$$+ 4·144 \frac{P_c}{\rho_c}\left(\frac{Z}{R}\right)^2 \gamma,$$

which becomes, by multiplication with the normalizing factor:

$$+ ·02155 \frac{n^2}{\omega^2} = + ·2074\, n^2.$$

The numerical value of this coefficient will, in the neighbourhood of the commensurability, change only slowly with changes in γ.

5. Equation (II) may be written:

$$\frac{dC}{dt} = -\frac{\partial H}{\partial \dot{C}}, \quad \frac{d\dot{C}}{dt} = \frac{\partial H}{\partial C};$$
$$\frac{dC'}{dt} = -\frac{\partial H}{\partial \dot{C}'}, \quad \frac{d\dot{C}'}{dt} = \frac{\partial H}{\partial C'},$$

where C and C' have been put for C_ω and $C_{\omega'}$.

New variables J and w, J' and w' are introduced by the equations:

$$C = \sqrt{\frac{2J}{n}} \cos w, \quad \dot{C} = \sqrt{2 J n} \sin w;$$
$$C' = \sqrt{\frac{2J'}{n'}} \cos w', \quad \dot{C}' = \sqrt{2 J' n'} \sin w'.$$

Since this is a canonical transformation, the new variables satisfy the equations:

$$\frac{dJ}{dt} = -\frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = \frac{\partial H}{\partial J};$$
$$\frac{dJ'}{dt} = -\frac{\partial H}{\partial w'}, \quad \frac{dw'}{dt} = \frac{\partial H}{\partial J'}.$$

Because $2n-n'$ is about zero, the terms in H with argument $2w-w'$ are the most important ones. This argument occurs only in a term given by the product C^2C', the coefficient of which has been computed in the preceding section. This product gives also terms with the arguments $2w+w'$ and w'; since these are not critical they have not been included. Thus H has been used in the form:

$$H = -nJ - n'J' + \cdot 1467\, Jn \sqrt{\frac{J'}{n'}} \cos(2w - w').$$

There exists a periodic solution for which w and w' change linearly with the time, such that $2w-w'$ is constant and J and J' are constant. If J and J' do not depend on the time, $\partial H/\partial w$ and $\partial H/\partial w'$ must be zero, which gives the condition:

$$\sin(2w - w') = 0.$$

This equation is satisfied for $2w-w'$ equal to zero or π, and $\cos(2w-w') = \pm 1$.

With: $H = -nJ - n'J' \pm \cdot 1467\, Jn \sqrt{\frac{J'}{n'}}$,

it is found that:

$$\frac{dw}{dt} = -n \pm \cdot 1467\, n \sqrt{\frac{J'}{n'}},$$
$$\frac{dw'}{dt} = -n' \pm \cdot 0734 \frac{Jn}{\sqrt{(J'n')}}.$$

The condition: $\frac{d}{dt}(2w - w') = 0$,

gives a quadratic equation for $\sqrt{(J'/n')}$, viz.

$$\left(\sqrt{\frac{J'}{n'}}\right)^2 \mp 3{\cdot}408 \frac{2n-n'}{n} \sqrt{\frac{J'}{n'}} - \frac{J}{4n'} = 0.$$

For a definite value of $2n-n'$ the two solutions found for $2w-w'$ equal to zero are the same as those for $2w-w'$ equal to π, because both $\sqrt{(J'/n')}$ and $\cos w'$ change sign, if $2w-w'$ is taken to be π instead of zero. Therefore it suffices to consider only one sign of the second term in the quadratic equation; the negative sign, corresponding to $2w-w'$ equal to zero, has been chosen.

Of the two solutions for $\sqrt{(J'/n')}$ only that one is used, which becomes zero for J equal to zero.

At a distance from the commensurability the root occurring in the expression for $\sqrt{(J'/n')}$ may be developed, which gives, if only the term of the first degree is included:

$$\sqrt{\frac{J'}{n'}} = 1{\cdot}704 \frac{2n-n'}{n}\left[1 - \sqrt{\left\{1 + \cdot 08609 \frac{J}{n'}\left(\frac{n}{2n-n'}\right)^2\right\}}\right] = -\cdot 0734 \frac{J}{n'} \frac{n}{2n-n'}.$$

Thus J' is seen to be proportional to J^2, in the same way as the coefficient found in the preceding paper [1]). On the other hand it is evident from the quadratic equation that for $2n$ approaching n', J' approaches the value $J/4$.

As the observations of RR Lyrae are treated of in section 7, the numerical data relating to this star may be used here:

$$\frac{J}{n} = \cdot 000308, \quad \frac{2n-n'}{n} = \frac{1}{67{\cdot}412}.$$

The second term in the square root is then equal to ·0607, so that development including the first-order term only, is a good approximation. It gives:

$$\sqrt{\frac{J'}{n'}} = -\cdot 000767.$$

The periodic solution derived here is only a particular solution. Adjacent ones may be found by considering variations of the variables [1]); this results in a differential equation of the second order, with constant coefficients, for the variation of the critical

1) *B.A.N.* **7**, p. 265, 1935.

1) C.f. Dr J. Woltjer Jr, *B.A.N.* **1**, p. 219, 1923.

argument. Thus $2w-w'$ is seen to change periodically with a frequency ν, given by:

$$\frac{\nu^2}{n^2} = \cdot 02150 \frac{J}{n'}\left(2 + \frac{J}{4J'}\right).$$

For increasing $|2n-n'|$ ν approaches $|2n-n'|$, but in the critical case of $|2n-n'|$ becoming equal to zero, ν reaches the finite value given by:

$$\frac{\nu^2}{n^2} = \cdot 03226 \frac{J}{n}.$$

For the numerical values used above $\nu/n = \cdot 01506$, whereas $|2n-n'|/n$ is equal to $\cdot 01483$.

6. In the previous section it has been shown how a solution may be found for all values of $2n-n'$, including zero, by treating the two characteristic oscillations together, taking into account the second-order terms from the beginning. However, at some distance from commensurability the equations can be solved more simply by successive approximations. This method, where the two characteristic oscillations are considered independently, has been used in the present section.

The equations are, up to second-order terms:

$$\frac{d^2C}{dt^2} + n^2 C = Q(C, C'\ldots),$$

$$\frac{d^2C'}{dt^2} + n'^2 C' = Q'(C, C'\ldots),$$

and analogous equations for C'' etc.; the functions Q are homogeneous and quadratic in the coefficients C.

The solutions for the equations with right-hand member equal to zero are:

$$C = A \cos (nt+\varphi),$$
$$C' = A' \cos (n't+\varphi'),$$
etc.

If a number of free oscillations are excited the light-curve will be variable, but if there is only one, the variation of light will be strictly periodic. A periodic solution of this kind has been sought in the earlier paper [1]); it has been derived by a different method, the displacement not being developed in terms of the functions s_ω. It is equivalent to the solution that would be found if the equations above for all coefficients C, C' etc. were solved, the right-hand members consisting only of the term with argument $2(nt+\varphi)$, given by the term proportional to C^2 in the functions Q.

If the solutions of the equations to the first order only are substituted in the quadratic terms, it is seen that for $2n$ close to n' the most important term in the right-hand member of the first of the equations above is the one with argument $(n'-n)\,t+\varphi'-\varphi$, and in the second one that with argument $2(nt+\varphi)$, since these give resonance. The equations for C'' etc. contain no such critical terms and have not been considered [2]). If only the above-mentioned terms are retained, the equations for C and C' become:

$$\frac{d^2C}{dt^2} + n^2 C = \cdot 2074\, AA'\, n^2 \cos \{ (n'-n)\, t + \varphi'-\varphi \},$$

$$\frac{d^2C'}{dt^2} + n'^2 C' = \cdot 1037\, A^2 n^2 \cos 2\, (nt+\varphi).$$

The solutions are:

$$C = A \cos (nt+\varphi) + \frac{\cdot 2074\, AA'\, n^2}{n^2-(n'-n)^2} \cos \{ (n'-n)\, t + \varphi'-\varphi \},$$

$$C' = A' \cos (n't+\varphi') + \frac{\cdot 1037\, A^2\, n^2}{n'^2-4\,n^2} \cos 2\, (nt+\varphi).$$

Hence:

$$\zeta = \left[A \cos (nt+\varphi) + \frac{\cdot 2074 AA'\, n^2}{n^2-(n'-n)^2} \cos \{ (n'-n)\, t + \varphi'-\varphi \}\right] s_\omega +$$
$$+ \left[A' \cos (n't+\varphi') + \frac{\cdot 1037\, A^2\, n^2}{n'^2-4\,n^2} \cos 2\, (nt+\varphi)\right] s_{\omega'}.$$

It is evident that for $2n$ close to n' the equations cannot be solved in this way, since the second and fourth terms then get small denominators; however, by the theory of periodic solutions as used in section 5, this difficulty is avoided, and it is found that for approximate commensurability the coefficient of the fourth term has a value of about $A/2\sqrt{2}$.

The argument of the second term in ζ may be written:

$$nt+\varphi+(n'-2n)\,t+\varphi'-2\varphi = nt+\varphi+\chi,$$

1) *B.A.N.* **7**, p. 265, 1935.

2) It may be stated once more explicitly that primes refer to the *third* characteristic vibration.

that of the third term:

$$2\,(nt+\varphi)+\chi;$$

χ has the frequency $n'-2n$. In the neighbourhood of the commensurability, however, the solution found in the preceding section must be used instead of the present one, and the argument χ is replaced by an argument with frequency ν; ν has the limiting value

$$\cdot 1796\,\sqrt{Jn}.$$

The displacement is strictly periodic if there is no free oscillation in C', so that A' is equal to zero; ζ then consists of the first and fourth terms only. The character of the velocity-curve depends on the relative values of the two coefficients, but also on the sign of the last term, which is here the same as that of $n'-2n$, since s_ω and $s_{\omega'}$ have the same sign at the outer boundary. It may be remarked here that the curves found before [1] would have had an entirely different shape if a slightly different value of γ had been chosen. Therefore the agreement found with the well-known empirical result that for the equivalent spectroscopic orbit the distance from node to periastron is about 90°, is more or less accidental.

7. In the variation of light of the cluster-type variable RR Lyrae two periods have been found, one of 0·56685 day and one of 38·21 days, or 67·41 times the short period [1]). The long period suggests interference between two approximately commensurable frequencies, and therefore the observations of this star have been interpreted with the aid of the present analysis.

Observers of the radial velocity have not mentioned the long period, but the radius may be expected to change with the same periods as the luminosity [2]). If 38·21 days corresponds to the frequency ν, this means that $n/\nu = 67\cdot4$, or $n/(2n-n') = \pm\,67\cdot4$, since it has been shown in section 5 that for RR Lyrae ν and $|2n-n'|$ are about equal. It has also been found there that the development of the root in the expression for $\sqrt{(J'/n')}$ gives a good approximation, so that the solutions of the preceding section may be used here. From the shape of the radial velocity-curve it is seen that the amplitudes of cos $(nt+\varphi)$ and cos $2(nt+\varphi)$ have opposite signs; therefore $n'-2n$ must be negative.

For the interpretation of SANFORD's velocity-curve, which is an average one, A' must be taken equal to zero; hence:

$$\frac{d\zeta}{dt} = -\,n\left\{A\,s_\omega \sin\,(nt+\varphi) + \frac{\cdot 2074\,A^2\,n^2}{n'^2-4n^2}\,s_{\omega'} \sin 2\,(nt+\varphi)\right\}.$$

The value of A is determined by equating the maximum of $-Rd\zeta/dt$ [2]) to the observed maximum radial velocity; this latter has been multiplied with 24/17, to account for the fact that the observed velocity is an average over the stellar disc, the intensity having been taken proportional to $2+3\cos\vartheta$. The resulting value of A reduces the equation to:

$$-\frac{1}{n}\frac{d\zeta}{dt} = \cdot 0365\,\sin\,(nt+\varphi) - \cdot 0491\,\sin 2\,(nt+\varphi).$$

This function has two maxima [3]), the principal one, equal to ·0764, at $nt+\varphi = 128°\cdot35$, and the secondary one of ·0253 at $nt+\varphi = 36°\cdot3$. SANFORD actually mentions the possibility of a secondary maximum [4]).

Moreover the periodic variation in the epoch of the light-curve exists, with a period of 38·21 days and an amplitude of about ·01 day. In order to account for this, another free oscillation must be brought in, and therefore A' must no longer be taken equal to zero. For the radial velocity the amplitude of this term will probably also be of the order of a few hundredths of a day, and for the purpose of making an estimate of A' it has been assumed to be $^d\cdot$01. Substitution of the boundary values of s_ω and $s_{\omega'}$ and of the value for A found above, gives for the additional terms in ζ:

$$\cdot 259\,A'\cos(nt+\varphi+\chi) \text{ and } 22\cdot 520\,A'\cos\{2\,(nt+\varphi)+\chi\},$$

so that evidently the first of these may be neglected as compared with the second one. Therefore:

$$\zeta = \cdot 0365\,\cos\,(nt+\varphi) - \cdot 0246\,\cos 2\,(nt+\varphi) + 22\cdot 520\,A'\,\cos\{2\,(nt+\varphi)+\chi\},$$

and, since χ is approximately constant during one period of ·56685 day:

$$-\frac{1}{n}\frac{d\zeta}{dt} = \cdot 0365\,\sin\,(nt+\varphi) - \cdot 0491\,\sin 2\,(nt+\varphi) + 45\cdot 040\,A'\,\sin\{2\,(nt+\varphi)+\chi\}.$$

1) l.c., p. 270.

2) For R the value given by EDDINGTON in *The Internal Constitution of the Stars*, Table 25, p. 182, has been used.

3) The curves, found in *B.A.N.* **7**, p. 270, show, for the larger values of a_1, the same characteristics.

4) *Ap. J.* **81**, p. 151, 1935.

1) Data about the variation of light have been taken from *B.A.N.* **6**, p. 215, 1932, by A. DE SITTER.

2) The radial velocity-curve used is the one given by R. F. SANFORD, *Ap. J.* **81**, p. 149, 1935.

The additional term changes both the time and the height of the maximum; the time is changed by a periodic term δt, given by:

$$n\,\delta t = 410{\cdot}01\, A' \cos(\chi + 256^\circ{\cdot}70).$$

Hence the value $A' = {\cdot}0002705$ results and:

$$-\frac{1}{n}\frac{d\zeta}{dt} = {\cdot}0248\, s_\omega \sin(nt+\varphi) - {\cdot}00218\, s_{\omega'} \sin 2(nt+\varphi) + {\cdot}0005410\, s_{\omega'} \sin\{2(nt+\varphi)+\chi\}.$$

It is seen that in the greater part of the star, where s_ω and $s_{\omega'}$ are of the same order of magnitude, the third term is very small compared with the first one, whereas at the boundary:

$$-\frac{1}{n}\frac{d\zeta}{dt} = {\cdot}0365 \sin(nt+\varphi) - {\cdot}0491 \sin 2(nt+\varphi) + {\cdot}0122 \sin\{2(nt+\varphi)+\chi\}.$$

EDDINGTON has pointed out that the reason why in general only one period is observed, probably is that the higher characteristic oscillations suffer a larger dissipation than the first one. Here it is seen that the third oscillation may, even though it is relatively small within the star, still give an observable effect in the radial velocity, as a consequence of the large increase of $s_{\omega'}$, as compared with s_ω, towards the boundary. The height of the principal maximum oscillates between ·0886 and ·0642, which is not incompatible with the observations.

8. *Summary.*

In this paper the terms of higher order in the theory of adiabatically pulsating stars have been investigated, and in particular those of the second order, for the case of approximate commensurability between two of the characteristic frequencies.

The displacement has been developed in terms of the amplitudes of the characteristic oscillations; it has been found possible to write the differential equations for the coefficients in this development, which are functions of the time only, in the canonical form (sections 3 and 5). In the case where $3-4/\gamma = 0{\cdot}4$, the first and third characteristic frequencies are approximately commensurable. For this value of $3-4/\gamma$ the solution has been determined in section 5 according to the method given by Dr WOLTJER, which is valid for all values of $2n-n'$, including zero; the solution can be studied close to and exactly in the commensurability. In section 6 the equations are solved by successive approximations, which procedure is only possible at some distance from the commensurability.

Finally the numerical results have been compared with the observed changes in radial velocity of RR Lyrae; the observations have been interpreted with the assumption that the long period observed in the variation of light must be attributed to the interference between the first and third characteristic oscillations. It has been found that the theoretical velocity-curve can represent the observations.

The long-periodic change of the epoch may be explained as a consequence of the free oscillation with the third characteristic frequency being excited; although the amplitude of this free oscillation is rather small in the interior of the star, it still gives an observable effect in the radial velocity.

With pleasure I express my sincere thanks to Dr WOLTJER for kindly having given me his guidance and advice during the work.