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Anomalous power law of quantum reversibility for classically regular dynamics

Ph. Jacquod, I. Adagideli and C. W. J. Beenakker

Instituut-Lorentz, Universiteit Leiden - P.O. Box 9506
2300 RA Leiden, The Netherlands

Abstract. – The Loschmidt Echo $M(t)$ (defined as the squared overlap of wave packets evolving with two slightly different Hamiltonians) is a measure of quantum reversibility. We investigate its behavior for classically quasi-integrable systems. A dominant regime emerges where $M(t) \propto t^{-\alpha}$ with $\alpha = 3d/2$ depending solely on the dimension d of the system. This power law decay is faster than the result $\propto t^{-d}$ for the decay of the overlap of classical phase space densities.

The search for quantum signatures of chaos has provided much insight into how classical dynamics manifests itself in quantum mechanics [1,2]. The basic question is how to determine from a system's quantum properties whether the classical limit of its dynamics is chaotic or regular. One very successful approach has been to look at the spectral statistics, in particular the distribution of level spacings [3]. An altogether different approach, advocated by Schack and Caves [4], has been to investigate the sensitivity of the quantum dynamics to perturbations of the Hamiltonian. This approach goes back to the early work of Peres [5] and has attracted new interest recently in connection with the study of decoherence and quantum reversibility [6–12].

The basic quantity in this approach is the so-called Loschmidt Echo, *i.e.* the fidelity

$$M(t) = |\langle \psi_0| \exp[iHt] \exp[-iH_0 t]|\psi_0\rangle|^2 \tag{1}$$

with which a narrow wavepacket ψ_0 can be reconstructed by inverting the dynamics after a time t with a perturbed Hamiltonian $H = H_0 + V$ [5,6]. (We set $\hbar = 1$.) The fidelity quantifies the sensitivity of the time-reversal operation to the uncertainty in the Hamiltonian, and thus provides for a measure of quantum reversibility.

To date, most investigations of $M(t)$ focused on classically chaotic Hamiltonians H and H_0 [6–10]. One notable exception is the original paper by Peres [5], who noted that the decay of $M(t)$ is slower in a regular system —but did not quantify it further. We will show in this article that in a regular system, and under certain randomness assumptions on the choice of the perturbation V (to be specified below), a dominant regime emerges where $M(t)$ has a power law decay $\propto t^{-3d/2}$, with an exponent depending solely on the dimension d of the system.

This power law decay establishes the higher degree of quantum reversibility of regular systems compared to chaotic ones, where $M(t)$ decays exponentially. This trend is as expected from classical reversibility (defined in terms of the decay of the overlap of classical phase space distributions [13]). However, we find that quantum mechanics plays a crucial role in regular systems by inducing a parametrically faster power law decay $\propto t^{-3d/2}$ than the classical one $\propto t^{-d}$.

We consider the generic situation of a regular or quasi-integrable H_0 and a perturbation potential V that has no common integral of motion with H_0. (By regular or quasi-integrable we mean systems with a phase space dominated by invariant tori.) This condition ensures that, classically, the perturbation has a component transverse to the invariant tori almost everywhere in phase space, and we will assume that this transverse component varies sufficiently rapidly along an unperturbed classical trajectory. Our investigation will moreover focus on a regime of sufficiently strong perturbation (defined below), where one expects a fast decay of the perturbation correlator. This regime is to be contrasted with the linear-response regime considered in ref. [11].

We follow the semiclassical approach of Jalabert and Pastawski [6]. We start from a Gaussian wavepacket $\psi_0(r_0') = (\pi\sigma^2)^{-d/4} \exp[ip_0 \cdot (r_0' - r_0) - |r_0' - r_0|^2/2\sigma^2]$ and approximate its time evolution by

$$\exp[-iHt]\psi_0(r) = \int dr_0' \sum_s K_s^H(r, r_0'; t)\psi_0(r_0'), \tag{2}$$

$$K_s^H(r, r_0'; t) = C_s^{1/2} \exp[iS_s^H(r, r_0'; t) - i\pi\mu_s/2]. \tag{3}$$

The semiclassical propagator is expressed as a sum over classical trajectories (labelled s) connecting r and r_0' in the time t. For each s, the partial propagator contains the action integral $S_s^H(r, r_0'; t)$ along s, a Maslov index μ_s (which will drop out), and the determinant C_s of the monodromy matrix. Since we consider a narrow initial wavepacket, we linearize the action in $r_0' - r_0$ and perform the integration over r_0'. After a stationary phase approximation, the semiclassical fidelity reads

$$M(t) = (4\pi\sigma^2)^d \left| \int dr \sum_s K_s^H(r, r_0; t)^* K_s^{H_0}(r, r_0; t) \exp\left[-\sigma^2|p_s - p_0|^2 \right] \right|^2, \tag{4}$$

with initial momentum $p_s = -\partial S_s/\partial r_0$.

Equations (2)-(4) are equally valid for regular and chaotic Hamiltonians, as long as semiclassics applies. Squaring the amplitude in eq. (4) leads to a double sum over classical paths s and s' and a double integration over coordinates r and r'. Accordingly, $M(t) = M^{(d)}(t) + M^{(nd)}(t)$ splits into diagonal ($s = s'$) and nondiagonal ($s \neq s'$) contributions. The diagonal contribution sensitively depends on whether H_0 is regular or chaotic. Reference [6] found that $M^{(d)}(t) \propto \exp[-\lambda t]$ for chaotic dynamics, with λ the Lyapunov exponent. We will show that the decay turns into a power law $M^{(d)}(t) \propto t^{-3d/2}$ for regular dynamics. The nondiagonal contribution, on the contrary, is insensitive to the nature of the classical dynamics (set by H_0), provided the perturbation Hamiltonian V has no common integral of motion with H_0. References [6,7] found that $M^{(nd)}(t) \propto \exp[-\Gamma t]$ for chaotic dynamics, and ref. [7] identified Γ with the golden-rule spreading width of an eigenstate of H_0 over the eigenbasis of H. (This golden-rule decay requires that Γ is larger than the level spacing Δ, but smaller than the bandwidth [7].) We will see that the same exponential decay of $M^{(nd)}(t)$ holds when H_0 is regular, so that $M^{(d)}(t)$ always dominates in the long-time limit. Consequently, the fidelity decays exponentially, $\propto \exp[-\min(\Gamma, \lambda)t]$ for chaotic systems, while for regular systems the decay is algebraic, $\propto t^{-3d/2}$, as is then set by the diagonal contribution. The golden-rule width Γ still determines the regime of validity of the power law decay via the condition $\Gamma > \Delta$.

Continuing from eq (4), and still following ref [6], we write $M(t)$ as

$$M(t) = \left(4\pi\sigma^2\right)^d \int d\mathbf{r} \int d\mathbf{r}' \sum_{s\,s'} C_s C_{s'} \exp\left[\imath\delta S_s(\mathbf{r},\mathbf{r}_0,t) - \imath\delta S_{s'}(\mathbf{r}',\mathbf{r}_0,t)\right] \times$$

$$\times \exp\left[-\sigma^2|\mathbf{p}_s - \mathbf{p}_0|^2 - \sigma^2|\mathbf{p}_{s'} - \mathbf{p}_0|^2\right], \tag{5}$$

with $\delta S_s(\mathbf{r},\mathbf{r}_0,t) = S_s^H(\mathbf{r},\mathbf{r}_0,t) - S_s^{H_0}(\mathbf{r},\mathbf{r}_0,t)$ Considering first the diagonal contribution $M^{(\mathrm{d})}(t)$, we set $s = s'$ and expand the phase difference as

$$\delta S_s(\mathbf{r},\mathbf{r}_0,t) - \delta S_s(\mathbf{r}',\mathbf{r}_0,t) = \int_0^t d\tilde{t}\,\nabla V[\mathbf{q}(\tilde{t})]\;\left(\mathbf{q}(\tilde{t}) - \mathbf{q}'(\tilde{t})\right) \tag{6}$$

The points $\mathbf{q}$ and $\mathbf{q}'$ lie on the classical path with $\mathbf{q}(t) = \mathbf{r}$, $\mathbf{q}'(t) = \mathbf{r}'$, and $\mathbf{q}(0) = \mathbf{q}'(0) = \mathbf{r}_0$ In a regular system, the distance between two initially close points increases linearly with time, $|\mathbf{q}(\tilde{t}) - \mathbf{q}'(\tilde{t})| \simeq (\tilde{t}/t)|\mathbf{r} - \mathbf{r}'|$ Here we depart from the exponential divergence $\propto \exp[\lambda(\tilde{t}-t)]$ assumed in ref [6] for chaotic dynamics

The spatial integrations and the sums over classical paths in eq (5) lead to the phase averaging

$$\exp[\imath\delta S_s - \imath\delta S_s'] \longrightarrow \langle\exp[\imath\delta S_s - \imath\delta S_s']\rangle \simeq \exp\left[-\frac{1}{2}\langle(\delta S_s - \delta S_s')^2\rangle\right] \tag{7}$$

The phase averaging is justified by our assumption that V varies rapidly along an unperturbed classical trajectory Since V and H_0 have no common integral of motion, we may expect a fast decay of the correlations,

$$\langle\partial_i V[\mathbf{q}(\tilde{t})]\partial_j V[\mathbf{q}(\tilde{t}')]\rangle = U\delta_{ij}\delta(\tilde{t} - \tilde{t}') \tag{8}$$

One then gets

$$M^{(\mathrm{d})}(t) = (4\pi\sigma^2)^d \int d\mathbf{r} \int d\mathbf{r}' \sum_s C_s^2 \exp\left[-\frac{1}{2}U\int_0^t d\tilde{t}(\tilde{t}/t)^2|\mathbf{r} - \mathbf{r}'|^2\right] \times$$

$$\times \exp\left[-2\sigma^2|\mathbf{p}_s - \mathbf{p}_0|^2\right]$$

$$= (4\pi\sigma^2)^d \int d\mathbf{r}_+ \int d\mathbf{r}_- \sum_s C_s^2 \exp\left[-\frac{1}{6}Ut\mathbf{r}_-^2\right]\exp\left[-2\sigma^2|\mathbf{p}_s - \mathbf{p}_0|^2\right] \tag{9}$$

The Gaussian integration over $\mathbf{r}_- \equiv \mathbf{r} - \mathbf{r}'$ ensures that $\mathbf{r} \approx \mathbf{r}'$, and hence $\mathbf{r}_+ \equiv (\mathbf{r}+\mathbf{r}')/2 \approx \mathbf{r}$ One C_s is then absorbed by a change of variable from $\mathbf{r}_+$ to $\mathbf{p}_s$, and the Gaussian integral over $\mathbf{r}_-$ gives a factor $\propto t^{-d/2}$ Finally, setting $C_s \approx t^{-d}$ as is the case in a regular system, we arrive at

$$M^{(\mathrm{d})}(t) \propto t^{-3d/2}, \tag{10}$$

which is the central result of this paper The power law (10) holds once the perturbation is strong enough to induce a golden-rule spreading of the eigenstates of H_0 over the eigenbasis of H (which is the range of validity [6,7] of the above semiclassical approach), and under the assumption that the perturbation potential varies rapidly along a classical trajectory of H_0 The decay exponent $3d/2$ is insensitive to the choice (8) of a δ-function force correlator Even a power law decaying correlator $\propto |\tilde{t} - \tilde{t}'|^{-\alpha}$ (with $\alpha \geq 1$) results in the same exponent as in eq (10)

The nondiagonal contribution ($s \neq s'$) to eq (5) is the same as in refs [6,7] The phase averaging can be performed separately for s and s' and one gets

$$\langle \exp[i\delta S_s] \rangle = \exp\left[-\frac{1}{2}\langle \delta S_s^2 \rangle \right] = \exp\left[-\frac{1}{2}\int_0^t d\tilde{t} \int_0^t d\tilde{t}' \langle V[q(\tilde{t})]V[q(\tilde{t}')] \rangle \right] \qquad (11)$$

The point $q(\tilde{t})$ lies on path s with $q(0) = r_0$ and $q(t) = r$ If V and H_0 have no common integral of motion, the correlator of V gives the golden-rule decay $\propto \exp[-\Gamma t]$, regardless of whether H_0 is chaotic or regular [14] We conclude that for regular systems, the fidelity is dominated by the algebraically decaying diagonal contribution

In order to check numerically the analytical result (10), we have studied the kicked top Hamiltonian [1]

$$H_0 = (\pi/2\tau)S_y + (K/2S)S_z^2 \sum_n \delta(t - n\tau), \qquad (12)$$

which describes a vector spin of conserved magnitude S, undergoing a free precession around the y-axis, which is periodically perturbed (period τ) by sudden rotations around the z-axis over an angle proportional to S_z Because S is conserved, H_0 is a one-dimensional Hamiltonian ($d = 1$), with a two-dimensional classical phase space consisting of the sphere of radius $S = 1$ The canonically conjugated variables are $(\varphi, \cos\theta)$, where θ and φ are spherical coordinates

The classical limit of the kicked top is given by the map [1]

$$\begin{cases} x_{n+1} = z_n \cos(Kx_n) + y_n \sin(Kx_n), \\ y_{n+1} = -z_n \sin(Kx_n) + y_n \cos(Kx_n), \\ z_{n+1} = -x_n, \end{cases} \qquad (13)$$

in the Cartesian coordinates $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and $z = \cos\theta$ Depending on the kicking strength K, the classical dynamics is regular, partially chaotic, or fully chaotic We consider a kicking strength $K = 1\,1$ for which the dynamics is regular for most of phase space We checked that our results are not sensitive to the value of K, as long as the dynamics remains regular

The quantum-mechanical time evolution after n periods is given by the n-th power of the Floquet operator

$$F_0 = \exp\left[-i(K/2S)S_z^2 \right] \exp\left[-i(\pi/2)S_y \right] \qquad (14)$$

We perturb the reversed-time evolution by a periodic rotation of constant angle around the x-axis, slightly delayed with respect to the kicks in H_0,

$$V = \phi S_i \sum_n \delta(t - n\tau - \epsilon) \qquad (15)$$

The corresponding Floquet operator is $F = \exp[-i\phi S_x]F_0$ We set $\tau = 1$ for ease of notation, and varied S between 250 and 1000 (both H and H_0 conserve the spin magnitude) We calculated the average decay $\overline{M}$ of $M(t = n) = |\langle\psi_0|(F^\dagger)^n F_0^n|\psi_0\rangle|^2$ taken over 50 to 200 initial Gaussian wavepackets ψ_0 of minimal spreading (coherent states)

In fig 1 we show the decay of $\overline{M}$ for $S = 1000$ and different perturbation strengths ϕ For weak perturbations, the decay of $\overline{M}$ is exponential, and not Gaussian as one would expect from first-order perturbation theory [5] The reason why we do not witness a Gaussian decay in that regime is that the perturbation operator V gives no first-order correction for low K
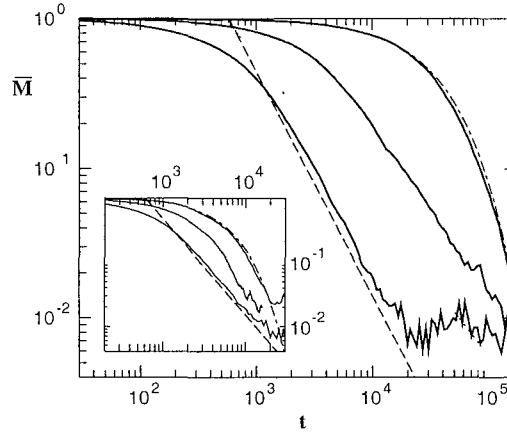
Fig. 1 – Decay of $\overline{M}$ for $S = 1000$, $K = 1\,1$, and $10^5\phi = 1.5$, 4.5, and 10 (thick solid lines from right to left). The crossover from exponential to power law decay is illustrated by the dot-dashed line $\propto \exp[-2\,56 \quad 10^{-5}t]$ and the dashed line $\propto t^{-3/2}$. The dotted line gives the classical decay $\propto t^{-1}$. Inset: decay of $\overline{M}$ for $K = 1\,1$, $\phi = 10^{-4}$, and $S = 250$, 500, and 1000 (solid lines from right to left). The dashed and dot-dashed lines indicate the power law $\propto t^{-3/2}$ and exponential $\propto \exp[-2 \cdot 10^{-4}t]$ decay, respectively These plots show that the $t^{-3/2}$ decay is reached either by increasing the perturbation strength ϕ at fixed spin magnitude S, or by increasing S at fixed ϕ.

Indeed, for $K = 1.1$, eigenfunctions of F_0 are still almost identical to eigenfunctions of S_y, so that diagonal matrix elements of $V \propto S_x$ vanish in this basis. Because of this, the local spectral density of states $\rho(\epsilon)$ for weak ϕ consists of a delta-function at zero energy plus an algebraically decaying tail [15]. In particular, the absence of first-order correction results in the absence of smearing of the delta-function at zero energy. Consequently, the decay of the fidelity is given by the Fourier transform of the tail of $\rho(\epsilon)$ [10]. We numerically obtained a decay $\rho(\epsilon) \propto (\epsilon^2 + \gamma^2/4)^{-1}$ with $\gamma \propto \phi^{1\,5}$ [16]. The resulting exponential decay $\propto \exp[-\gamma t]$ of the fidelity differs from the golden-rule decay $\propto \exp[-\Gamma t]$ with $\Gamma \propto \phi^2$.

As ϕ increases, and looking back at fig. 1, the decay of $\overline{M}$ turns into the predicted power law $\propto t^{-3/2}$, which prevails as soon as one enters the golden-rule regime, *i.e.* for $\Gamma/\Delta \approx \phi^2 S^3 \geq 1$ [7]. One, therefore, expects the power law decay to appear as S is increased at fixed ϕ, which is indeed observed in the inset to fig. 1.

We checked that these results are not sensitive to our choice of Hamiltonian, by replacing S_y in eq. (15) with S_z^2 (this is the model used in ref. [11]) and also by studying a kicked rotator as an alternative model to the kicked top. These numerical results all give clear confirmation of the power law decay (10).

It is instructive to contrast these results for the decay of the overlap of quantum wave functions with the decay of the overlap of classical phase space distributions, a "classical fidelity" problem that has recently been investigated [9,11,13]. We assume that the two phase space distributions ρ_0 and ρ are initially identical and evolve according to the Liouville equation of motion corresponding to the classical map (13) for two different Hamiltonians H_0 and H. We consider regular dynamics and ask for the decay of the normalized phase space overlap,

$$M_c(t) = \int \mathrm{d}x \int \mathrm{d}p \rho_0(x, p; t)\rho(x, p; t)/N_\rho, \tag{16}$$

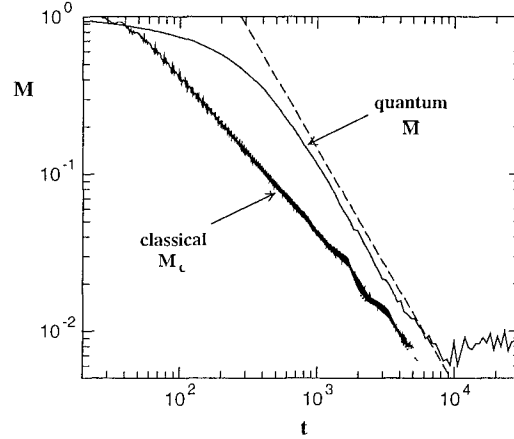where $N_\rho = (\int \mathrm{d}x \int \mathrm{d}p \rho_0)^{1/2}(\int \mathrm{d}x \int \mathrm{d}p \rho)^{1/2}$.

Fig 2 – Decay of the quantum fidelity $\overline{M}$ for $S = 1000$, compared to the decay of the average overlap M_c of classical phase space distributions, both for the kicked top with $K = 1.1$ and $\phi = 1\,7 \cdot 10^{-4}$. The initial classical distribution extends over a volume $\sigma = 10^{-3}$ of phase space, corresponding to one Planck cell for $S = 1000$ The dotted and dashed lines give the classical and quantum power law decays $\propto t^{-1}$ and $\propto t^{-3/2}$, respectively.

We have found above that a factor $\propto t^{-d/2}$ in the decay of the quantum fidelity $M(t) \propto t^{-3d/2}$ originates from the action phase difference and is thus of purely quantum origin. One therefore expects a slower classical decay $M_c(t) \propto C_s \propto t^{-d}$. In fig. 2 we show the decay of the averaged $\overline{M}_c$ taken over 10^4 initial points within a narrow volume of phase space $\sigma \equiv \sin\theta\delta\theta\delta\varphi$, for $K = 1.1$ and $\phi = 1.7 \cdot 10^{-4}$. The decay is $\overline{M}_c \propto t^{-1}$, and clearly differs from the quantum decay $\propto t^{-3/2}$.

The power law decay prevails for classically weak perturbations, for which the center of mass of ρ and ρ_0 stay close together. (This is required by the stationary phase condition leading to eq. (4).) Keeping σ fixed, and increasing the perturbation strength ϕ, the invariant tori of H_0 start to differ significantly from those of H on the resolution scale σ, giving a threshold $\phi_c \approx \sigma$. Above ϕ_c, the distance between the center of mass of ρ_0 and ρ increases with time $\propto t$ and one expects a much faster decay $M_c(t) \propto \exp[-\text{const} \times t^2]$ for classical Gaussian phase space distributions [13]. Quantum-mechanically, $\sigma = 1/S$ (the effective Planck constant) and the threshold translates into $\phi_c \sim 1/S$, coinciding with the upper boundary of the golden-rule regime. As long as one stays in that regime, the perturbation will affect the phase in eq. (7), and result in the anomalous power law decay $\propto t^{-3d/2}$.

In conclusion, our investigations of the Loschmidt Echo (1) in the generic regime of classically quasi-integrable dynamics show that its decay is dominated by the power law $M(t) \propto t^{-\alpha}$. While from purely classical considerations one expects an exponent $\alpha_c = d$, we semiclassically obtain an anomalous exponent $\alpha = 3d/2$, under the assumption that the perturbation potential varies sufficiently rapidly along an unperturbed classical trajectory. While this latter assumption is generically satisfied (i.e. for almost any V) in the case of a chaotic H_0, it restricts the choice of the perturbation potential V for a regular H_0. We corroborated the anomalous power law decay by numerical simulations on a standard model of quantum chaos, and therefore conclude that the choice of V is not too restricted. The power law decay is to be contrasted with the exponential decay found for chaotic systems.

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[16] In most circumstances, one expects $\rho(c)$ to have Lorentzian tails with $\gamma = \Gamma \propto \phi^2$, corresponding to the golden-rule regime In several instances however, one finds $\gamma \propto \phi^a$, with $1 \leq a \leq 2$ [15]